



Government of Karnataka

MATHEMATICS

Textbook for Class IX

Part - II



0962



राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्
NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING

Karnataka Textbook Society (R.)

6th Cross, Malleshwaram, Bengaluru - 560003

Constitution of India

Part IV A (Article 51 A)

Fundamental Duties

It shall be the duty of every citizen of India —

- (a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;
- (b) to cherish and follow the noble ideals which inspired our national struggle for freedom;
- (c) to uphold and protect the sovereignty, unity and integrity of India;
- (d) to defend the country and render national service when called upon to do so;
- (e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;
- (f) to value and preserve the rich heritage of our composite culture;
- (g) to protect and improve the natural environment including forests, lakes, rivers, wildlife and to have compassion for living creatures;
- (h) to develop the scientific temper, humanism and the spirit of inquiry and reform;
- (i) to safeguard public property and to abjure violence;
- (j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;
- * (k) who is a parent or guardian, to provide opportunities for education to his child or, as the case may be, ward between the age of six and fourteen years.

Note: The Article 51A containing Fundamental Duties was inserted by the Constitution (42nd Amendment) Act, 1976 (with effect from 3 January 1977).

* (k) was inserted by the Constitution (86th Amendment) Act, 2002 (with effect from 1 April 2010).

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CHAPTER 7

TRIANGLES

7.1 Introduction

You have studied about triangles and their various properties in your earlier classes. You know that a closed figure formed by three intersecting lines is called a triangle. ('Tri' means 'three'). A triangle has three sides, three angles and three vertices. For example, in triangle ABC, denoted as $\triangle ABC$ (see Fig. 7.1); AB, BC, CA are the three sides, $\angle A$, $\angle B$, $\angle C$ are the three angles and A, B, C are three vertices.

In Chapter 6, you have also studied some properties of triangles. In this chapter, you will study in details about the congruence of triangles, rules of congruence, some more properties of triangles and inequalities in a triangle. You have already verified most of these properties in earlier classes. We will now prove some of them.

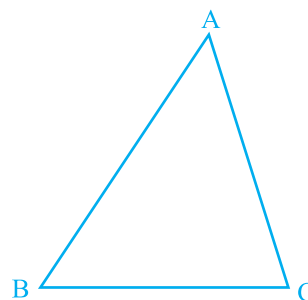


Fig. 7.1

7.2 Congruence of Triangles

You must have observed that two copies of your photographs of the same size are identical. Similarly, two bangles of the same size, two ATM cards issued by the same bank are identical. You may recall that on placing a one rupee coin on another minted in the same year, they cover each other completely.

Do you remember what such figures are called? Indeed they are called **congruent figures** ('congruent' means equal in all respects or figures whose shapes and sizes are both the same).

Now, draw two circles of the same radius and place one on the other. What do you observe? They cover each other completely and we call them as congruent circles.

Repeat this activity by placing one square on the other with sides of the same measure (see Fig. 7.2) or by placing two equilateral triangles of equal sides on each other. You will observe that the squares are congruent to each other and so are the equilateral triangles.

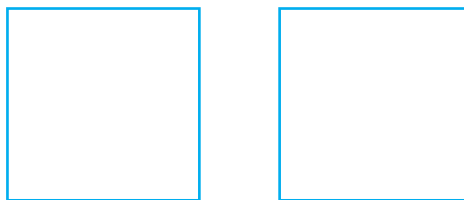


Fig. 7.2

You may wonder why we are studying congruence. You all must have seen the ice tray in your refrigerator. Observe that the moulds for making ice are all congruent. The cast used for moulding in the tray also has congruent depressions (may be all are rectangular or all circular or all triangular). So, whenever identical objects have to be produced, the concept of congruence is used in making the cast.

Sometimes, you may find it difficult to replace the refill in your pen by a new one and this is so when the new refill is not of the same size as the one you want to remove. Obviously, if the two refills are identical or congruent, the new refill fits.

So, you can find numerous examples where congruence of objects is applied in daily life situations.

Can you think of some more examples of congruent figures?

Now, which of the following figures are not congruent to the square in Fig 7.3 (i) :

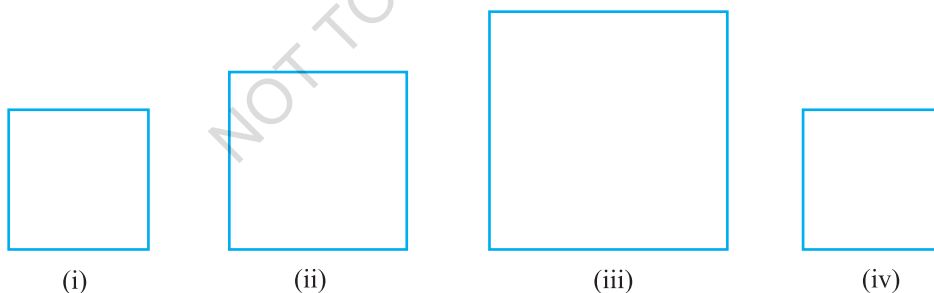


Fig. 7.3

The large squares in Fig. 7.3 (ii) and (iii) are obviously not congruent to the one in Fig 7.3 (i), but the square in Fig 7.3 (iv) is congruent to the one given in Fig 7.3 (i).

Let us now discuss the congruence of two triangles.

You already know that two triangles are congruent if the sides and angles of one triangle are equal to the corresponding sides and angles of the other triangle.

Now, which of the triangles given below are congruent to triangle ABC in Fig. 7.4 (i)?

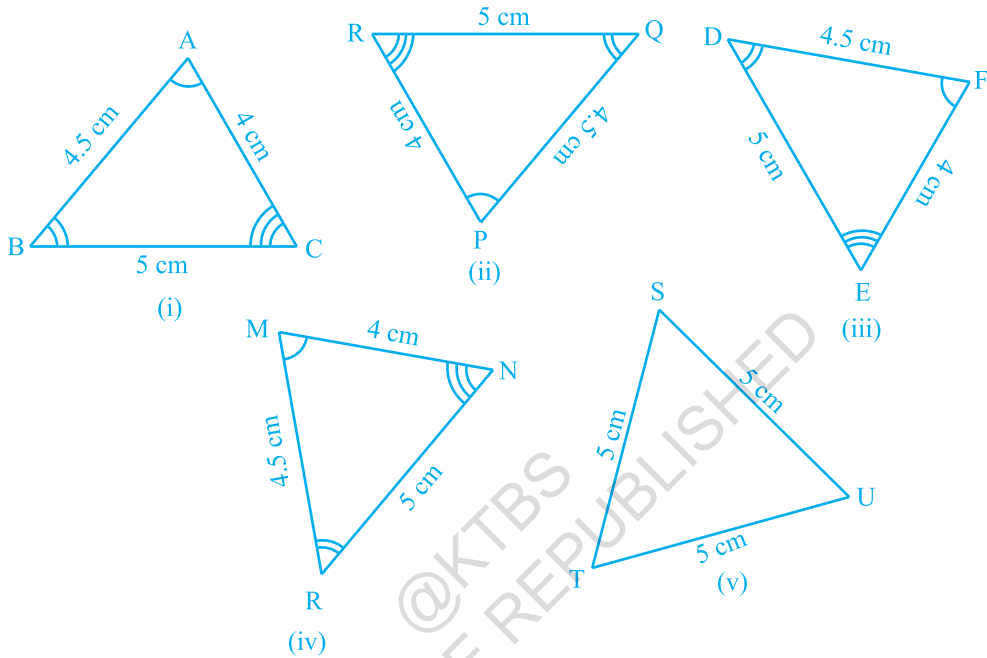


Fig. 7.4

Cut out each of these triangles from Fig. 7.4 (ii) to (v) and turn them around and try to cover $\triangle ABC$. Observe that triangles in Fig. 7.4 (ii), (iii) and (iv) are congruent to $\triangle ABC$ while $\triangle TSU$ of Fig. 7.4 (v) is not congruent to $\triangle ABC$.

If $\triangle PQR$ is congruent to $\triangle ABC$, we write $\triangle PQR \cong \triangle ABC$.

Notice that when $\triangle PQR \cong \triangle ABC$, then sides of $\triangle PQR$ fall on corresponding equal sides of $\triangle ABC$ and so is the case for the angles.

That is, PQ covers AB, QR covers BC and RP covers CA; $\angle P$ covers $\angle A$, $\angle Q$ covers $\angle B$ and $\angle R$ covers $\angle C$. Also, there is a one-one correspondence between the vertices. That is, P corresponds to A, Q to B, R to C and so on which is written as

$$P \leftrightarrow A, Q \leftrightarrow B, R \leftrightarrow C$$

Note that under this correspondence, $\triangle PQR \cong \triangle ABC$; but it will not be correct to write $\triangle QRP \cong \triangle ABC$.

Similarly, for Fig. 7.4 (iii),

$FD \leftrightarrow AB$, $DE \leftrightarrow BC$ and $EF \leftrightarrow CA$

and $F \leftrightarrow A$, $D \leftrightarrow B$ and $E \leftrightarrow C$

So, $\Delta FDE \cong \Delta ABC$ but writing $\Delta DEF \cong \Delta ABC$ is not correct.

Give the correspondence between the triangle in Fig. 7.4 (iv) and ΔABC .

So, it is necessary to write the correspondence of vertices correctly for writing of congruence of triangles in symbolic form.

Note that in **congruent triangles corresponding parts are equal** and we write in short 'CPCT' for *corresponding parts of congruent triangles*.

7.3 Criteria for Congruence of Triangles

In earlier classes, you have learnt four criteria for congruence of triangles. Let us recall them.

Draw two triangles with one side 3 cm. Are these triangles congruent? Observe that they are not congruent (see Fig. 7.5).

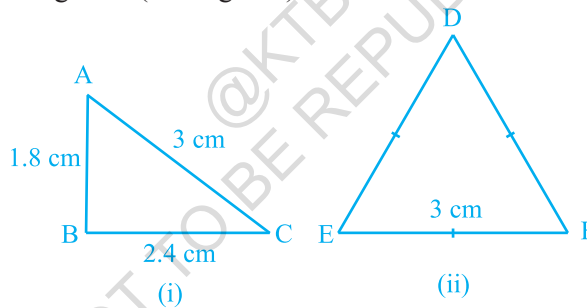


Fig. 7.5

Now, draw two triangles with one side 4 cm and one angle 50° (see Fig. 7.6). Are they congruent?

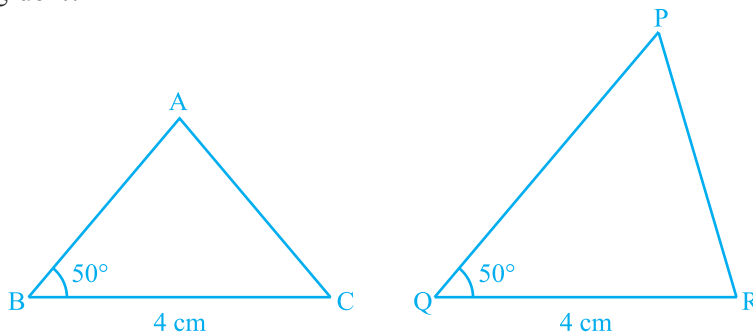


Fig. 7.6

See that these two triangles are not congruent.

Repeat this activity with some more pairs of triangles.

So, equality of one pair of sides or one pair of sides and one pair of angles is not sufficient to give us congruent triangles.

What would happen if the other pair of arms (sides) of the equal angles are also equal?

In Fig 7.7, $BC = QR$, $\angle B = \angle Q$ and also, $AB = PQ$. Now, what can you say about congruence of $\triangle ABC$ and $\triangle PQR$?

Recall from your earlier classes that, in this case, the two triangles are congruent. Verify this for $\triangle ABC$ and $\triangle PQR$ in Fig. 7.7.

Repeat this activity with other pairs of triangles. Do you observe that the equality of two sides and the included angle is enough for the congruence of triangles? Yes, it is enough.

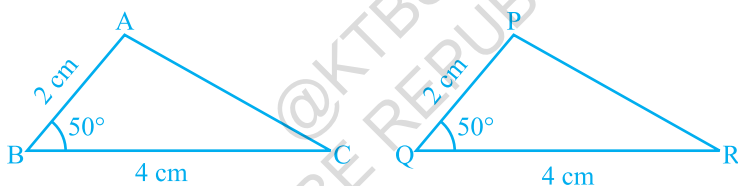


Fig. 7.7

This is the first criterion for congruence of triangles.

Axiom 7.1 (SAS congruence rule) : *Two triangles are congruent if two sides and the included angle of one triangle are equal to the two sides and the included angle of the other triangle.*

This result cannot be proved with the help of previously known results and so it is accepted true as an axiom (see Appendix 1).

Let us now take some examples.

Example 1 : In Fig. 7.8, $OA = OB$ and $OD = OC$. Show that

- (i) $\triangle AOD \cong \triangle BOC$ and (ii) $AD \parallel BC$.

Solution : (i) You may observe that in $\triangle AOD$ and $\triangle BOC$,

$$\left. \begin{array}{l} OA = OB \\ OD = OC \end{array} \right\} \text{ (Given)}$$

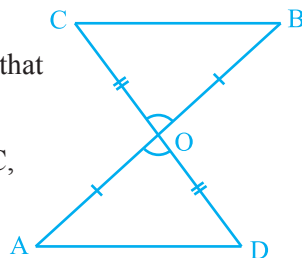


Fig. 7.8

Also, since $\angle AOD$ and $\angle BOC$ form a pair of vertically opposite angles, we have

$$\angle AOD = \angle BOC.$$

So, $\triangle AOD \cong \triangle BOC$ (by the SAS congruence rule)

(ii) In congruent triangles AOD and BOC , the other corresponding parts are also equal.

So, $\angle OAD = \angle OBC$ and these form a pair of alternate angles for line segments AD and BC .

Therefore, $AD \parallel BC$.

Example 2 : AB is a line segment and line l is its perpendicular bisector. If a point P lies on l , show that P is equidistant from A and B .

Solution : Line $l \perp AB$ and passes through C which is the mid-point of AB (see Fig. 7.9). You have to show that $PA = PB$. Consider $\triangle PCA$ and $\triangle PCB$.

We have $AC = BC$ (C is the mid-point of AB)

$$\angle PCA = \angle PCB = 90^\circ \quad (\text{Given})$$

$$PC = PC \quad (\text{Common})$$

So, $\triangle PCA \cong \triangle PCB$ (SAS rule)

and so, $PA = PB$, as they are corresponding sides of congruent triangles.

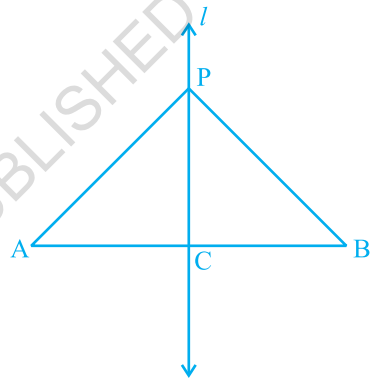


Fig. 7.9

Now, let us construct two triangles, whose sides are 4 cm and 5 cm and one of the angles is 50° and this angle is not included in between the equal sides (see Fig. 7.10). Are the two triangles congruent?

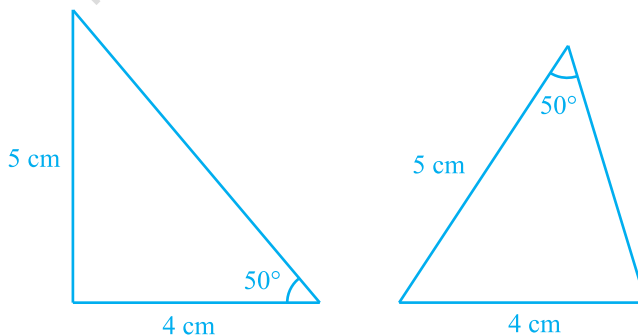


Fig. 7.10

Notice that the two triangles are not congruent.

Repeat this activity with more pairs of triangles. You will observe that for triangles to be congruent, it is very important that the equal angles are included between the pairs of equal sides.

So, *SAS congruence rule holds* but not ASS or SSA rule.

Next, try to construct the two triangles in which two angles are 60° and 45° and the side included between these angles is 4 cm (see Fig. 7.11).

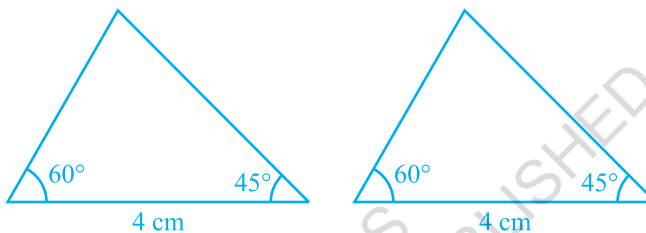


Fig. 7.11

Cut out these triangles and place one triangle on the other. What do you observe? See that one triangle covers the other completely; that is, the two triangles are congruent. Repeat this activity with more pairs of triangles. You will observe that equality of two angles and the included side is sufficient for congruence of triangles.

This result is the **Angle-Side-Angle** criterion for congruence and is written as **ASA** criterion. You have verified this criterion in earlier classes, but let us state and prove this result.

Since this result can be proved, it is called a theorem and to prove it, we use the SAS axiom for congruence.

Theorem 7.1 (ASA congruence rule) : *Two triangles are congruent if two angles and the included side of one triangle are equal to two angles and the included side of other triangle.*

Proof : We are given two triangles ABC and DEF in which:

$$\angle B = \angle E, \angle C = \angle F$$

and

$$BC = EF$$

We need to prove that $\triangle ABC \cong \triangle DEF$

For proving the congruence of the two triangles see that three cases arise.

Case (i) : Let $AB = DE$ (see Fig. 7.12).

Now what do you observe? You may observe that

$$AB = DE \quad (\text{Assumed})$$

$$\angle B = \angle E \quad (\text{Given})$$

$$BC = EF \quad (\text{Given})$$

So, $\triangle ABC \cong \triangle DEF$ (By SAS rule)

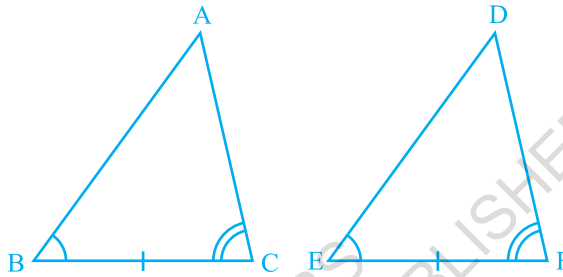


Fig. 7.12

Case (ii) : Let if possible $AB > DE$. So, we can take a point P on AB such that $PB = DE$. Now consider $\triangle PBC$ and $\triangle DEF$ (see Fig. 7.13).

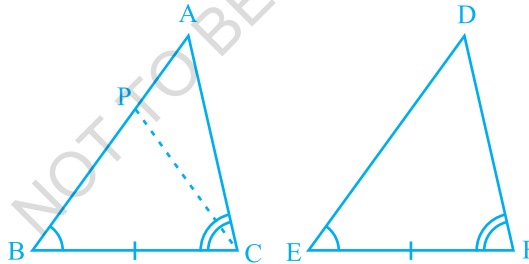


Fig. 7.13

Observe that in $\triangle PBC$ and $\triangle DEF$,

$$PB = DE \quad (\text{By construction})$$

$$\angle B = \angle E \quad (\text{Given})$$

$$BC = EF \quad (\text{Given})$$

So, we can conclude that:

$\triangle PBC \cong \triangle DEF$, by the SAS axiom for congruence.

Since the triangles are congruent, their corresponding parts will be equal.

So, $\angle PCB = \angle DFE$

But, we are given that

$$\angle ACB = \angle DFE$$

So, $\angle ACB = \angle PCB$

Is this possible?

This is possible only if P coincides with A.

or, $BA = ED$

So, $\triangle ABC \cong \triangle DEF$ (by SAS axiom)

Case (iii) : If $AB < DE$, we can choose a point M on DE such that $ME = AB$ and repeating the arguments as given in Case (ii), we can conclude that $AB = DE$ and so, $\triangle ABC \cong \triangle DEF$.

Suppose, now in two triangles two pairs of angles and one pair of corresponding sides are equal but the side is not included between the corresponding equal pairs of angles. Are the triangles still congruent? You will observe that they are congruent. Can you reason out why?

You know that the sum of the three angles of a triangle is 180° . So if two pairs of angles are equal, the third pair is also equal ($180^\circ - \text{sum of equal angles}$).

So, *two triangles are congruent if any two pairs of angles and one pair of corresponding sides are equal*. We may call it as the **AAS Congruence Rule**.

Now let us perform the following activity :

Draw triangles with angles 40° , 50° and 90° . How many such triangles can you draw?

In fact, you can draw as many triangles as you want with different lengths of sides (see Fig. 7.14).

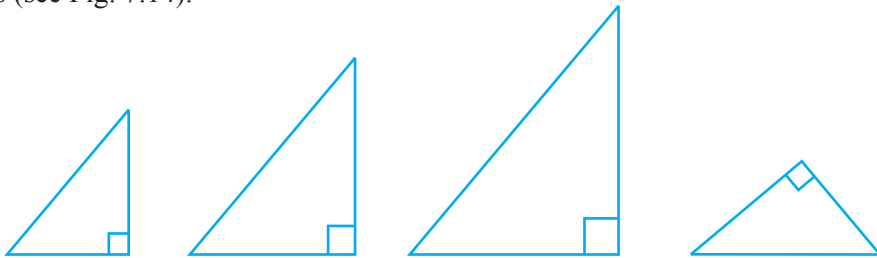


Fig. 7.14

Observe that the triangles may or may not be congruent to each other.

So, equality of three angles is not sufficient for congruence of triangles. Therefore, for congruence of triangles out of three equal parts, one has to be a side.

Let us now take some more examples.

Example 3 : Line-segment AB is parallel to another line-segment CD. O is the mid-point of AD (see Fig. 7.15). Show that (i) $\triangle AOB \cong \triangle DOC$ (ii) O is also the mid-point of BC.

Solution : (i) Consider $\triangle AOB$ and $\triangle DOC$.

$$\angle ABO = \angle DCO$$

(Alternate angles as $AB \parallel CD$
and BC is the transversal)

$$\angle AOB = \angle DOC$$

(Vertically opposite angles)

$$OA = OD \quad (\text{Given})$$

Therefore, $\triangle AOB \cong \triangle DOC$ (AAS rule)

$$(ii) \quad OB = OC \quad (\text{CPCT})$$

So, O is the mid-point of BC.

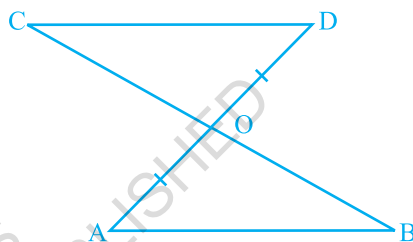


Fig. 7.15

EXERCISE 7.1

1. In quadrilateral ACBD,

$AC = AD$ and AB bisects $\angle A$
(see Fig. 7.16). Show that $\triangle ABC \cong \triangle ABD$.

What can you say about BC and BD?

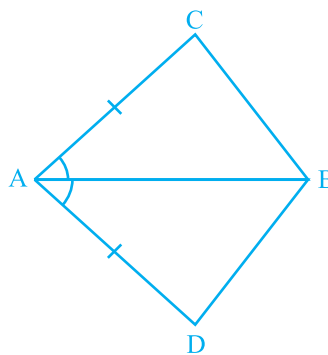


Fig. 7.16

2. ABCD is a quadrilateral in which $AD = BC$ and $\angle DAB = \angle CBA$ (see Fig. 7.17). Prove that
- $\triangle ABD \cong \triangle BAC$
 - $BD = AC$
 - $\angle ABD = \angle BAC$.

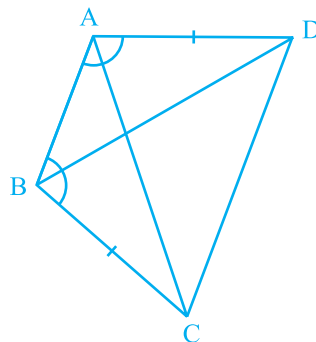


Fig. 7.17

3. AD and BC are equal perpendiculars to a line segment AB (see Fig. 7.18). Show that CD bisects AB.

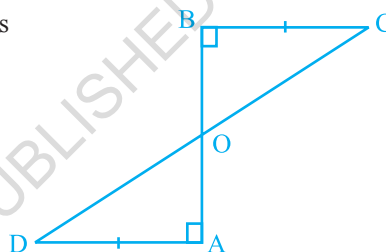


Fig. 7.18

4. l and m are two parallel lines intersected by another pair of parallel lines p and q (see Fig. 7.19). Show that $\triangle ABC \cong \triangle CDA$.

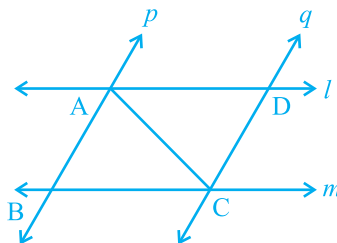


Fig. 7.19

5. Line l is the bisector of an angle $\angle A$ and B is any point on l . BP and BQ are perpendiculars from B to the arms of $\angle A$ (see Fig. 7.20). Show that:
- $\triangle APB \cong \triangle AQB$
 - $BP = BQ$ or B is equidistant from the arms of $\angle A$.

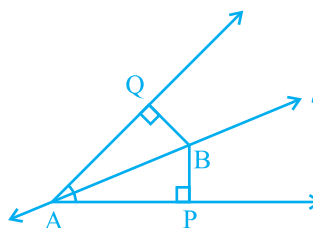


Fig. 7.20

6. In Fig. 7.21, $AC = AE$, $AB = AD$ and $\angle BAD = \angle EAC$. Show that $BC = DE$.

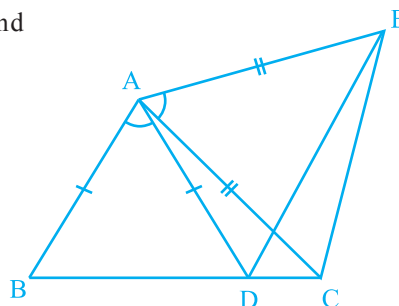


Fig. 7.21

7. AB is a line segment and P is its mid-point. D and E are points on the same side of AB such that $\angle BAD = \angle ABE$ and $\angle EPA = \angle DPB$ (see Fig. 7.22). Show that
- $\triangle DAP \cong \triangle EBP$
 - $AD = BE$

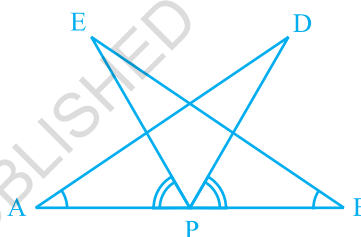


Fig. 7.22

8. In right triangle ABC , right angled at C , M is the mid-point of hypotenuse AB . C is joined to M and produced to a point D such that $DM = CM$. Point D is joined to point B (see Fig. 7.23). Show that:
- $\triangle AMC \cong \triangle BMD$
 - $\angle DBC$ is a right angle.
 - $\triangle DBC \cong \triangle ACB$
 - $CM = \frac{1}{2} AB$

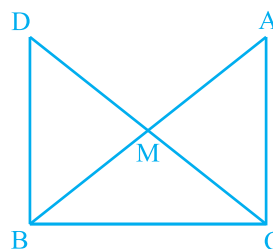


Fig. 7.23

7.4 Some Properties of a Triangle

In the above section you have studied two criteria for congruence of triangles. Let us now apply these results to study some properties related to a triangle whose two sides are equal.

Perform the activity given below:

Construct a triangle in which two sides are equal, say each equal to 3.5 cm and the third side equal to 5 cm (see Fig. 7.24). You have done such constructions in earlier classes.

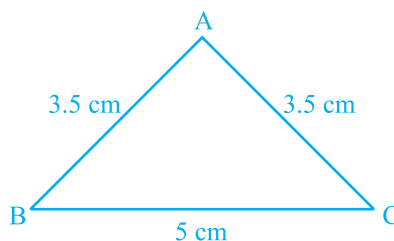


Fig. 7.24

A triangle in which two sides are equal is called an **isosceles triangle**. So, $\triangle ABC$ of Fig. 7.24 is an isosceles triangle with $AB = AC$.

Now, measure $\angle B$ and $\angle C$. What do you observe?

Repeat this activity with other isosceles triangles with different sides.

You may observe that in each such triangle, the angles opposite to the equal sides are equal.

This is a very important result and is indeed true for any isosceles triangle. It can be proved as shown below.

Theorem 7.2 : *Angles opposite to equal sides of an isosceles triangle are equal.*

This result can be proved in many ways. One of the proofs is given here.

Proof : We are given an isosceles triangle ABC in which $AB = AC$. We need to prove that $\angle B = \angle C$.

Let us draw the bisector of $\angle A$ and let D be the point of intersection of this bisector of $\angle A$ and BC (see Fig. 7.25).

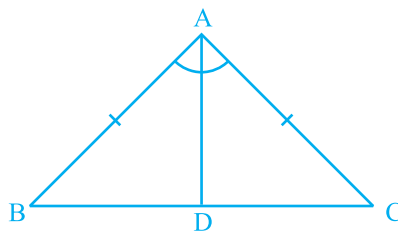


Fig. 7.25

In $\triangle BAD$ and $\triangle CAD$,

$$AB = AC \quad \text{(Given)}$$

$$\angle BAD = \angle CAD \quad \text{(By construction)}$$

$$AD = AD \quad \text{(Common)}$$

$$\text{So, } \triangle BAD \cong \triangle CAD \quad \text{(By SAS rule)}$$

So, $\angle ABD = \angle ACD$, since they are corresponding angles of congruent triangles.

$$\text{So, } \angle B = \angle C$$

Is the converse also true? That is:

If two angles of any triangle are equal, can we conclude that the sides opposite to them are also equal?

Perform the following activity.

Construct a triangle ABC with BC of any length and $\angle B = \angle C = 50^\circ$. Draw the bisector of $\angle A$ and let it intersect BC at D (see Fig. 7.26).

Cut out the triangle from the sheet of paper and fold it along AD so that vertex C falls on vertex B.

What can you say about sides AC and AB?

Observe that AC covers AB completely

So, $AC = AB$

Repeat this activity with some more triangles. Each time you will observe that the sides opposite to equal angles are equal. So we have the following:

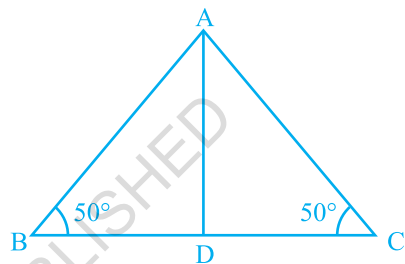


Fig. 7.26

Theorem 7.3 : *The sides opposite to equal angles of a triangle are equal.*

This is the converse of Theorem 7.2.

You can prove this theorem by ASA congruence rule.

Let us take some examples to apply these results.

Example 4 : In $\triangle ABC$, the bisector AD of $\angle A$ is perpendicular to side BC (see Fig. 7.27). Show that $AB = AC$ and $\triangle ABC$ is isosceles.

Solution : In $\triangle ABD$ and $\triangle ACD$,

$$\angle BAD = \angle CAD \quad (\text{Given})$$

$$AD = AD \quad (\text{Common})$$

$$\angle ADB = \angle ADC = 90^\circ \quad (\text{Given})$$

$$\text{So, } \triangle ABD \cong \triangle ACD \quad (\text{ASA rule})$$

$$\text{So, } AB = AC \quad (\text{CPCT})$$

or, $\triangle ABC$ is an isosceles triangle.

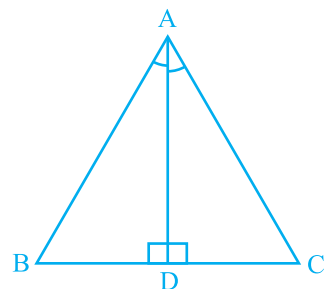


Fig. 7.27

Example 5 : E and F are respectively the mid-points of equal sides AB and AC of $\triangle ABC$ (see Fig. 7.28). Show that $BF = CE$.

Solution : In $\triangle ABF$ and $\triangle ACE$,

$$AB = AC \quad (\text{Given})$$

$$\angle A = \angle A \quad (\text{Common})$$

$$AF = AE \quad (\text{Halves of equal sides})$$

$$\text{So, } \triangle ABF \cong \triangle ACE \quad (\text{SAS rule})$$

$$\text{Therefore, } BF = CE \quad (\text{CPCT})$$

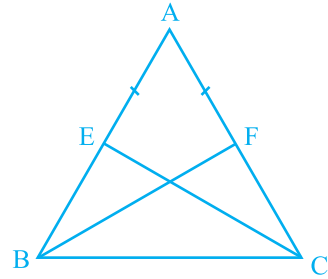


Fig. 7.28

Example 6 : In an isosceles triangle ABC with $AB = AC$, D and E are points on BC such that $BE = CD$ (see Fig. 7.29). Show that $AD = AE$.

Solution : In $\triangle ABD$ and $\triangle ACE$,

$$AB = AC \quad (\text{Given}) \quad (1)$$

$$\angle B = \angle C \quad (\text{Angles opposite to equal sides}) \quad (2)$$

$$\text{Also, } BE = CD$$

$$\text{So, } BE - DE = CD - DE$$

$$\text{That is, } BD = CE \quad (3)$$

$$\text{So, } \triangle ABD \cong \triangle ACE \quad (\text{Using (1), (2), (3) and SAS rule}).$$

$$\text{This gives } AD = AE \quad (\text{CPCT})$$

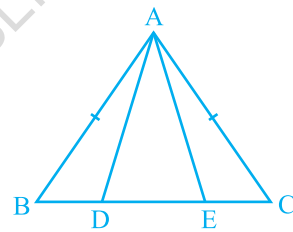


Fig. 7.29

EXERCISE 7.2

- In an isosceles triangle ABC, with $AB = AC$, the bisectors of $\angle B$ and $\angle C$ intersect each other at O. Join A to O. Show that :
 - $OB = OC$
 - AO bisects $\angle A$
- In $\triangle ABC$, AD is the perpendicular bisector of BC (see Fig. 7.30). Show that $\triangle ABC$ is an isosceles triangle in which $AB = AC$.

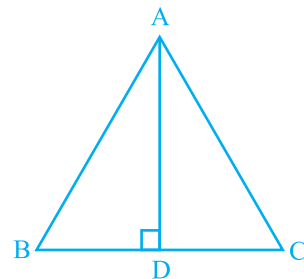


Fig. 7.30

3. ABC is an isosceles triangle in which altitudes BE and CF are drawn to equal sides AC and AB respectively (see Fig. 7.31). Show that these altitudes are equal.

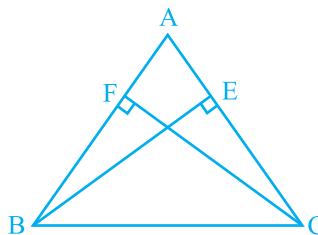


Fig. 7.31

4. ABC is a triangle in which altitudes BE and CF to sides AC and AB are equal (see Fig. 7.32). Show that

- (i) $\triangle ABE \cong \triangle ACF$
- (ii) $AB = AC$, i.e., ABC is an isosceles triangle.

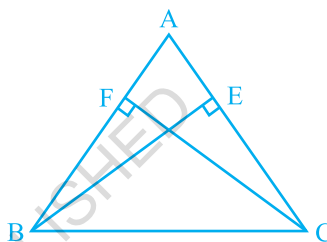


Fig. 7.32

5. ABC and DBC are two isosceles triangles on the same base BC (see Fig. 7.33). Show that $\angle ABD = \angle ACD$.

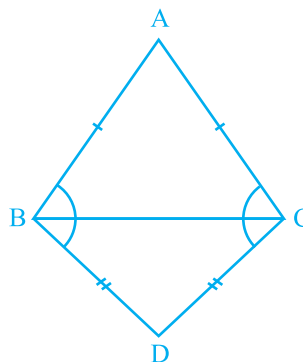


Fig. 7.33

6. $\triangle ABC$ is an isosceles triangle in which $AB = AC$. Side BA is produced to D such that $AD = AB$ (see Fig. 7.34). Show that $\angle BCD$ is a right angle.
7. ABC is a right angled triangle in which $\angle A = 90^\circ$ and $AB = AC$. Find $\angle B$ and $\angle C$.
8. Show that the angles of an equilateral triangle are 60° each.

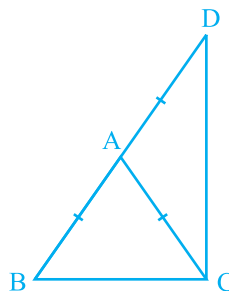


Fig. 7.34

7.5 Some More Criteria for Congruence of Triangles

You have seen earlier in this chapter that equality of three angles of one triangle to three angles of the other is not sufficient for the congruence of the two triangles. You may wonder whether equality of three sides of one triangle to three sides of another triangle is enough for congruence of the two triangles. You have already verified in earlier classes that this is indeed true.

To be sure, construct two triangles with sides 4 cm, 3.5 cm and 4.5 cm (see Fig. 7.35). Cut them out and place them on each other. What do you observe? They cover each other completely, if the equal sides are placed on each other. So, the triangles are congruent.

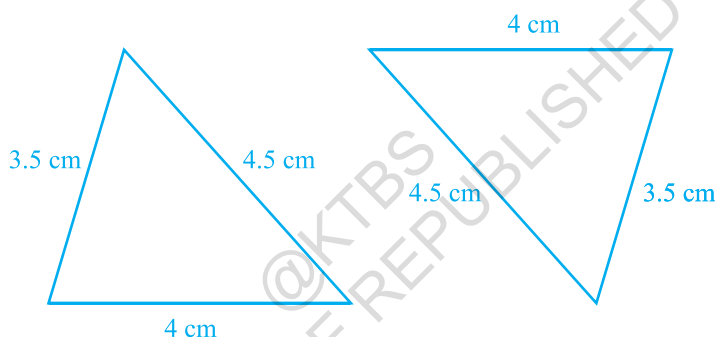


Fig. 7.35

Repeat this activity with some more triangles. We arrive at another rule for congruence.

Theorem 7.4 (SSS congruence rule) : *If three sides of one triangle are equal to the three sides of another triangle, then the two triangles are congruent.*

This theorem can be proved using a suitable construction.

You have already seen that in the SAS congruence rule, the pair of equal angles has to be the included angle between the pairs of corresponding pair of equal sides and if this is not so, the two triangles may not be congruent.

Perform this activity:

Construct two right angled triangles with hypotenuse equal to 5 cm and one side equal to 4 cm each (see Fig. 7.36).

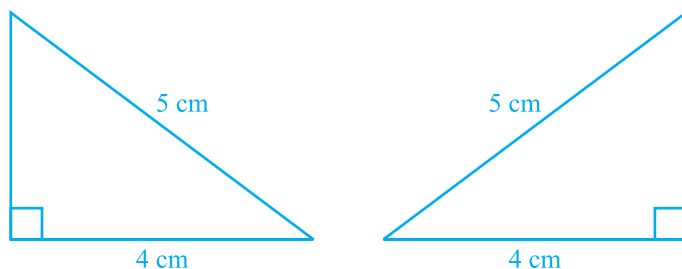


Fig. 7.36

Cut them out and place one triangle over the other with equal side placed on each other. Turn the triangles, if necessary. What do you observe?

The two triangles cover each other completely and so they are congruent. Repeat this activity with other pairs of right triangles. What do you observe?

You will find that two right triangles are congruent if one pair of sides and the hypotenuse are equal. You have verified this in earlier classes.

Note that, the right angle is **not** the included angle in this case.

So, you arrive at the following congruence rule:

Theorem 7.5 (RHS congruence rule) : *If in two right triangles the hypotenuse and one side of one triangle are equal to the hypotenuse and one side of the other triangle, then the two triangles are congruent.*

Note that RHS stands for **Right angle - Hypotenuse - Side**.

Let us now take some examples.

Example 7 : AB is a line-segment. P and Q are points on opposite sides of AB such that each of them is equidistant from the points A and B (see Fig. 7.37). Show that the line PQ is the perpendicular bisector of AB.

Solution : You are given that $PA = PB$ and $QA = QB$ and you are to show that $PQ \perp AB$ and PQ bisects AB. Let PQ intersect AB at C.

Can you think of two congruent triangles in this figure?

Let us take $\triangle PAQ$ and $\triangle PBQ$.

In these triangles,

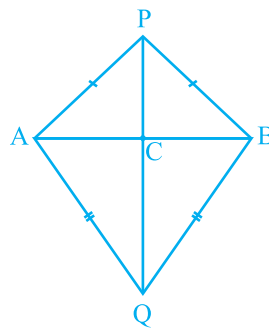


Fig. 7.37

$AP = BP$ (Given)
 $AQ = BQ$ (Given)
 $PQ = PQ$ (Common)
 So, $\triangle PAQ \cong \triangle PBQ$ (SSS rule)
 Therefore, $\angle APQ = \angle BPQ$ (CPCT).

Now let us consider $\triangle PAC$ and $\triangle PBC$.

You have : $AP = BP$ (Given)
 $\angle APC = \angle BPC$ ($\angle APQ = \angle BPQ$ proved above)
 $PC = PC$ (Common)
 So, $\triangle PAC \cong \triangle PBC$ (SAS rule)
 Therefore, $AC = BC$ (CPCT) (1)
 and $\angle ACP = \angle BCP$ (CPCT)
 Also, $\angle ACP + \angle BCP = 180^\circ$ (Linear pair)
 So, $2\angle ACP = 180^\circ$
 or, $\angle ACP = 90^\circ$ (2)

From (1) and (2), you can easily conclude that PQ is the perpendicular bisector of AB.

[Note that, without showing the congruence of $\triangle PAQ$ and $\triangle PBQ$, you cannot show that $\triangle PAC \cong \triangle PBC$ even though $AP = BP$ (Given)

$PC = PC$ (Common)
 and $\angle PAC = \angle PBC$ (Angles opposite to equal sides in $\triangle APB$)

It is because these results give us SSA rule which is not always valid or true for congruence of triangles. Also the angle is not included between the equal pairs of sides.]

Let us take some more examples.

Example 8 : P is a point equidistant from two lines l and m intersecting at point A (see Fig. 7.38). Show that the line AP bisects the angle between them.

Solution : You are given that lines l and m intersect each other at A. Let $PB \perp l$, $PC \perp m$. It is given that $PB = PC$.

You are to show that $\angle PAB = \angle PAC$.

Let us consider $\triangle PAB$ and $\triangle PAC$. In these two triangles,

$$PB = PC \quad (\text{Given})$$

$$\angle PBA = \angle PCA = 90^\circ \quad (\text{Given})$$

$$PA = PA \quad (\text{Common})$$

$$\text{So, } \triangle PAB \cong \triangle PAC \quad (\text{RHS rule})$$

$$\text{So, } \angle PAB = \angle PAC \quad (\text{CPCT})$$

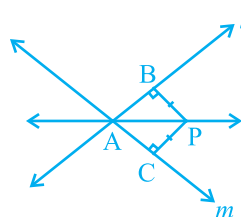


Fig. 7.38

Note that this result is the converse of the result proved in Q.5 of Exercise 7.1.

EXERCISE 7.3

1. $\triangle ABC$ and $\triangle DBC$ are two isosceles triangles on the same base BC and vertices A and D are on the same side of BC (see Fig. 7.39). If AD is extended to intersect BC at P , show that

- (i) $\triangle ABD \cong \triangle ACD$
- (ii) $\triangle ABP \cong \triangle ACP$
- (iii) AP bisects $\angle A$ as well as $\angle D$.
- (iv) AP is the perpendicular bisector of BC .

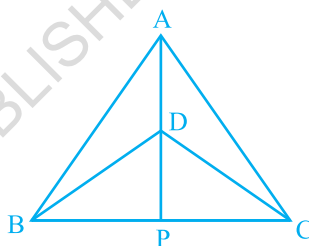


Fig. 7.39

2. AD is an altitude of an isosceles triangle ABC in which $AB = AC$. Show that
 - (i) AD bisects BC
 - (ii) AD bisects $\angle A$.

3. Two sides AB and BC and median AM of one triangle ABC are respectively equal to sides PQ and QR and median PN of $\triangle PQR$ (see Fig. 7.40). Show that:

- (i) $\triangle ABM \cong \triangle PQN$
- (ii) $\triangle ABC \cong \triangle PQR$

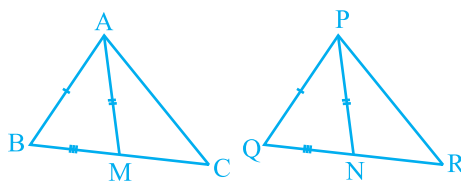


Fig. 7.40

4. BE and CF are two equal altitudes of a triangle ABC . Using RHS congruence rule, prove that the triangle ABC is isosceles.
5. ABC is an isosceles triangle with $AB = AC$. Draw $AP \perp BC$ to show that $\angle B = \angle C$.

7.6 Summary

In this chapter, you have studied the following points :

1. Two figures are congruent, if they are of the same shape and of the same size.
2. Two circles of the same radii are congruent.
3. Two squares of the same sides are congruent.
4. If two triangles ABC and PQR are congruent under the correspondence $A \leftrightarrow P$, $B \leftrightarrow Q$ and $C \leftrightarrow R$, then symbolically, it is expressed as $\Delta ABC \cong \Delta PQR$.
5. If two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle, then the two triangles are congruent (SAS Congruence Rule).
6. If two angles and the included side of one triangle are equal to two angles and the included side of the other triangle, then the two triangles are congruent (ASA Congruence Rule).
7. If two angles and one side of one triangle are equal to two angles and the corresponding side of the other triangle, then the two triangles are congruent (AAS Congruence Rule).
8. Angles opposite to equal sides of a triangle are equal.
9. Sides opposite to equal angles of a triangle are equal.
10. Each angle of an equilateral triangle is of 60° .
11. If three sides of one triangle are equal to three sides of the other triangle, then the two triangles are congruent (SSS Congruence Rule).
12. If in two right triangles, hypotenuse and one side of a triangle are equal to the hypotenuse and one side of other triangle, then the two triangles are congruent (RHS Congruence Rule).



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CHAPTER 8

QUADRILATERALS

8.1 Properties of a Parallelogram

You have already studied quadrilaterals and their types in Class VIII. A quadrilateral has four sides, four angles and four vertices. A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel.

Let us perform an activity.

Cut out a parallelogram from a sheet of paper and cut it along a diagonal (see Fig. 8.1). You obtain two triangles. What can you say about these triangles?

Place one triangle over the other. Turn one around, if necessary. What do you observe?

Observe that the two triangles are congruent to each other.

Repeat this activity with some more parallelograms. Each time you will observe that each diagonal divides the parallelogram into two congruent triangles.

Let us now prove this result.

Theorem 8.1 : *A diagonal of a parallelogram divides it into two congruent triangles.*

Proof : Let ABCD be a parallelogram and AC be a diagonal (see Fig. 8.2). Observe that the diagonal AC divides parallelogram ABCD into two triangles, namely, $\triangle ABC$ and $\triangle CDA$. We need to prove that these triangles are congruent.

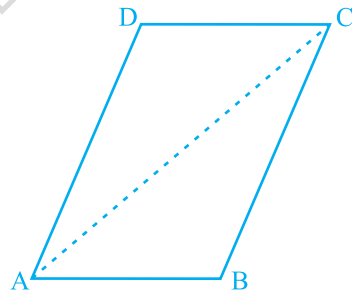


Fig. 8.1

In $\triangle ABC$ and $\triangle CDA$, note that $BC \parallel AD$ and AC is a transversal.

So, $\angle BCA = \angle DAC$ (Pair of alternate angles)

Also, $AB \parallel DC$ and AC is a transversal.

So, $\angle BAC = \angle DCA$ (Pair of alternate angles)

and $AC = CA$ (Common)

So, $\triangle ABC \cong \triangle CDA$ (ASA rule)

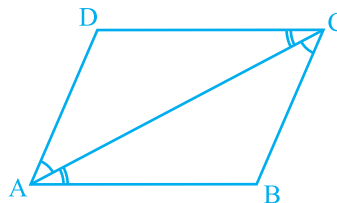


Fig. 8.2

or, diagonal AC divides parallelogram $ABCD$ into two congruent triangles ABC and CDA .

Now, measure the opposite sides of parallelogram $ABCD$. What do you observe?

You will find that $AB = DC$ and $AD = BC$.

This is another property of a parallelogram stated below:

Theorem 8.2 : *In a parallelogram, opposite sides are equal.*

You have already proved that a diagonal divides the parallelogram into two congruent triangles; so what can you say about the corresponding parts say, the corresponding sides? They are equal.

So, $AB = DC$ and $AD = BC$

Now what is the converse of this result? You already know that whatever is given in a theorem, the same is to be proved in the converse and whatever is proved in the theorem it is given in the converse. Thus, Theorem 8.2 can be stated as given below :

If a quadrilateral is a parallelogram, then each pair of its opposite sides is equal. So its converse is :

Theorem 8.3 : *If each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram.*

Can you reason out why?

Let sides AB and CD of the quadrilateral $ABCD$ be equal and also $AD = BC$ (see Fig. 8.3). Draw diagonal AC .

Clearly, $\triangle ABC \cong \triangle CDA$ (Why?)

So, $\angle BAC = \angle DCA$

and $\angle BCA = \angle DAC$ (Why?)

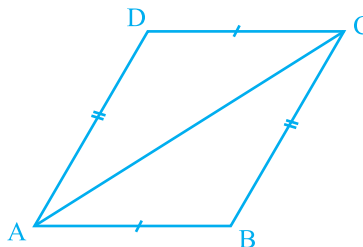


Fig. 8.3

Can you now say that $ABCD$ is a parallelogram? Why?

You have just seen that in a parallelogram each pair of opposite sides is equal and conversely if each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram. Can we conclude the same result for the pairs of opposite angles?

Draw a parallelogram and measure its angles. What do you observe?

Each pair of opposite angles is equal.

Repeat this with some more parallelograms. We arrive at yet another result as given below.

Theorem 8.4 : *In a parallelogram, opposite angles are equal.*

Now, is the converse of this result also true? Yes. Using the angle sum property of a quadrilateral and the results of parallel lines intersected by a transversal, we can see that the converse is also true. So, we have the following theorem :

Theorem 8.5 : *If in a quadrilateral, each pair of opposite angles is equal, then it is a parallelogram.*

There is yet another property of a parallelogram. Let us study the same. Draw a parallelogram ABCD and draw both its diagonals intersecting at the point O (see Fig. 8.4).

Measure the lengths of OA, OB, OC and OD.

What do you observe? You will observe that

$$OA = OC \quad \text{and} \quad OB = OD.$$

or, O is the mid-point of both the diagonals.

Repeat this activity with some more parallelograms.

Each time you will find that O is the mid-point of both the diagonals.

So, we have the following theorem :

Theorem 8.6 : *The diagonals of a parallelogram bisect each other.*

Now, what would happen, if in a quadrilateral the diagonals bisect each other? Will it be a parallelogram? Indeed this is true.

This result is the converse of the result of Theorem 8.6. It is given below:

Theorem 8.7 : *If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.*

You can reason out this result as follows:

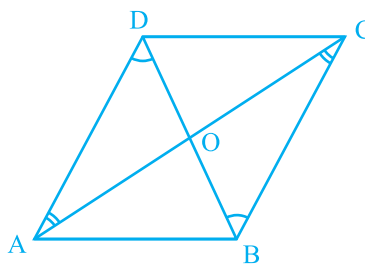


Fig. 8.4

Note that in Fig. 8.5, it is given that $OA = OC$ and $OB = OD$.

So, $\triangle AOB \cong \triangle COD$ (Why?)

Therefore, $\angle ABO = \angle CDO$ (Why?)

From this, we get $AB \parallel CD$

Similarly, $BC \parallel AD$

Therefore ABCD is a parallelogram.

Let us now take some examples.

Example 1 : Show that each angle of a rectangle is a right angle.

Solution : Let us recall what a rectangle is.

A rectangle is a parallelogram in which one angle is a right angle.

Let ABCD be a rectangle in which $\angle A = 90^\circ$.

We have to show that $\angle B = \angle C = \angle D = 90^\circ$

We have, $AD \parallel BC$ and AB is a transversal (see Fig. 8.6).

So, $\angle A + \angle B = 180^\circ$ (Interior angles on the same side of the transversal)

But, $\angle A = 90^\circ$

So, $\angle B = 180^\circ - \angle A = 180^\circ - 90^\circ = 90^\circ$

Now, $\angle C = \angle A$ and $\angle D = \angle B$

(Opposite angles of the parallelogram)

So, $\angle C = 90^\circ$ and $\angle D = 90^\circ$.

Therefore, each of the angles of a rectangle is a right angle.

Example 2 : Show that the diagonals of a rhombus are perpendicular to each other.

Solution : Consider the rhombus ABCD (see Fig. 8.7).

You know that $AB = BC = CD = DA$ (Why?)

Now, in $\triangle AOD$ and $\triangle COD$,

$OA = OC$ (Diagonals of a parallelogram bisect each other)

$OD = OD$ (Common)

$AD = CD$

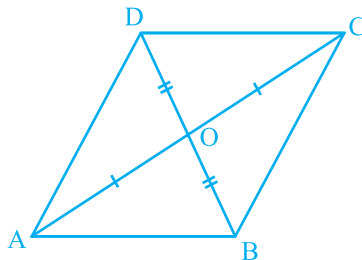


Fig. 8.5

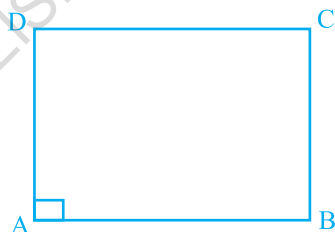


Fig. 8.6

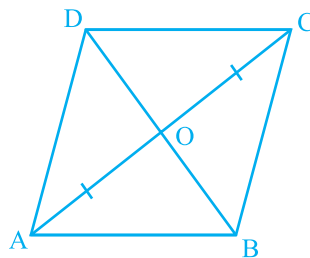


Fig. 8.7

Therefore, $\triangle AOD \cong \triangle COD$
(SSS congruence rule)

This gives, $\angle AOD = \angle COD$ (CPCT)

But, $\angle AOD + \angle COD = 180^\circ$ (Linear pair)

So, $2\angle AOD = 180^\circ$

or, $\angle AOD = 90^\circ$

So, the diagonals of a rhombus are perpendicular to each other.

Example 3 : ABC is an isosceles triangle in which $AB = AC$. AD bisects exterior angle PAC and $CD \parallel AB$ (see Fig. 8.8). Show that

(i) $\angle DAC = \angle BCA$ and (ii) ABCD is a parallelogram.

Solution : (i) $\triangle ABC$ is isosceles in which $AB = AC$ (Given)

So, $\angle ABC = \angle ACB$ (Angles opposite to equal sides)

Also, $\angle PAC = \angle ABC + \angle ACB$
(Exterior angle of a triangle)

or, $\angle PAC = 2\angle ACB$ (1)

Now, AD bisects $\angle PAC$.

So, $\angle PAC = 2\angle DAC$ (2)

Therefore,

$2\angle DAC = 2\angle ACB$ [From (1) and (2)]

or, $\angle DAC = \angle ACB$

(ii) Now, these equal angles form a pair of alternate angles when line segments BC and AD are intersected by a transversal AC.

So, $BC \parallel AD$

Also, $BA \parallel CD$ (Given)

Now, both pairs of opposite sides of quadrilateral ABCD are parallel.

So, ABCD is a parallelogram.

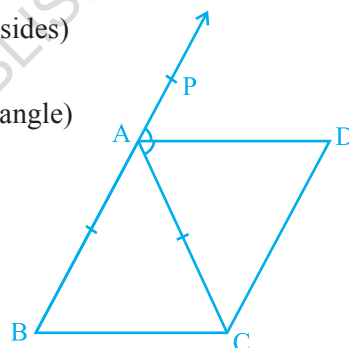


Fig. 8.8

Example 4 : Two parallel lines l and m are intersected by a transversal p (see Fig. 8.9). Show that the quadrilateral formed by the bisectors of interior angles is a rectangle.

Solution : It is given that $PS \parallel QR$ and transversal p intersects them at points A and C respectively.

The bisectors of $\angle PAC$ and $\angle ACQ$ intersect at B and bisectors of $\angle ACR$ and $\angle SAC$ intersect at D.

We are to show that quadrilateral ABCD is a rectangle.

Now, $\angle PAC = \angle ACR$

(Alternate angles as $l \parallel m$ and p is a transversal)

So, $\frac{1}{2} \angle PAC = \frac{1}{2} \angle ACR$

i.e., $\angle BAC = \angle ACD$

These form a pair of alternate angles for lines AB and DC with AC as transversal and they are equal also.

So, $AB \parallel DC$

Similarly, $BC \parallel AD$ (Considering $\angle ACB$ and $\angle CAD$)

Therefore, quadrilateral ABCD is a parallelogram.

Also, $\angle PAC + \angle CAS = 180^\circ$ (Linear pair)

So, $\frac{1}{2} \angle PAC + \frac{1}{2} \angle CAS = \frac{1}{2} \times 180^\circ = 90^\circ$

or, $\angle BAC + \angle CAD = 90^\circ$

or, $\angle BAD = 90^\circ$

So, ABCD is a parallelogram in which one angle is 90° .

Therefore, ABCD is a rectangle.

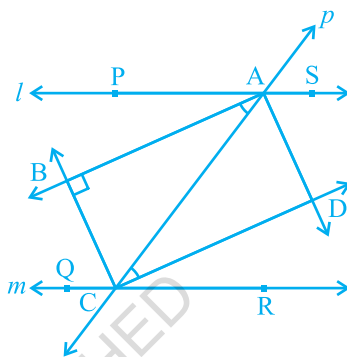


Fig. 8.9

Example 5 : Show that the bisectors of angles of a parallelogram form a rectangle.

Solution : Let P, Q, R and S be the points of intersection of the bisectors of $\angle A$ and $\angle B$, $\angle B$ and $\angle C$, $\angle C$ and $\angle D$, and $\angle D$ and $\angle A$ respectively of parallelogram ABCD (see Fig. 8.10).

In $\triangle ASD$, what do you observe?

Since DS bisects $\angle D$ and AS bisects $\angle A$, therefore,

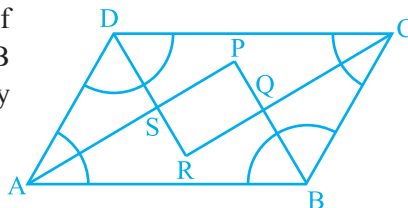


Fig. 8.10

$$\begin{aligned}
 \angle DAS + \angle ADS &= \frac{1}{2} \angle A + \frac{1}{2} \angle D \\
 &= \frac{1}{2} (\angle A + \angle D) \\
 &= \frac{1}{2} \times 180^\circ \quad (\angle A \text{ and } \angle D \text{ are interior angles} \\
 &\quad \text{on the same side of the transversal}) \\
 &= 90^\circ
 \end{aligned}$$

Also, $\angle DAS + \angle ADS + \angle DSA = 180^\circ$ (Angle sum property of a triangle)

or, $90^\circ + \angle DSA = 180^\circ$

or, $\angle DSA = 90^\circ$

So, $\angle PSR = 90^\circ$ (Being vertically opposite to $\angle DSA$)

Similarly, it can be shown that $\angle APB = 90^\circ$ or $\angle SPQ = 90^\circ$ (as it was shown for $\angle DSA$). Similarly, $\angle PQR = 90^\circ$ and $\angle SRQ = 90^\circ$.

So, PQRS is a quadrilateral in which all angles are right angles.

Can we conclude that it is a rectangle? Let us examine. We have shown that $\angle PSR = \angle PQR = 90^\circ$ and $\angle SPQ = \angle SRQ = 90^\circ$. So both pairs of opposite angles are equal.

Therefore, PQRS is a parallelogram in which one angle (in fact all angles) is 90° and so, PQRS is a rectangle.

EXERCISE 8.1

1. If the diagonals of a parallelogram are equal, then show that it is a rectangle.
2. Show that the diagonals of a square are equal and bisect each other at right angles.
3. Diagonal AC of a parallelogram ABCD bisects $\angle A$ (see Fig. 8.11). Show that
 - (i) it bisects $\angle C$ also,
 - (ii) ABCD is a rhombus.
4. ABCD is a rectangle in which diagonal AC bisects $\angle A$ as well as $\angle C$. Show that: (i) ABCD is a square
(ii) diagonal BD bisects $\angle B$ as well as $\angle D$.

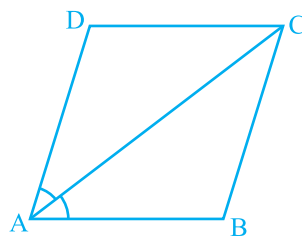


Fig. 8.11

5. In parallelogram ABCD, two points P and Q are taken on diagonal BD such that $DP = BQ$ (see Fig. 8.12). Show that:

- (i) $\triangle APD \cong \triangle CQB$
- (ii) $AP = CQ$
- (iii) $\triangle AQB \cong \triangle CPD$
- (iv) $AQ = CP$
- (v) APCQ is a parallelogram

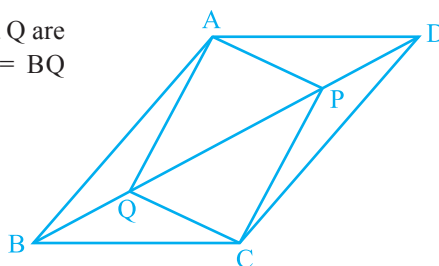


Fig. 8.12

6. ABCD is a parallelogram and AP and CQ are perpendiculars from vertices A and C on diagonal BD (see Fig. 8.13). Show that

- (i) $\triangle APB \cong \triangle CQD$
- (ii) $AP = CQ$

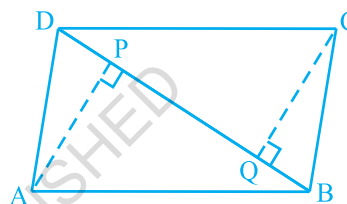


Fig. 8.13

7. ABCD is a trapezium in which $AB \parallel CD$ and $AD = BC$ (see Fig. 8.14). Show that

- (i) $\angle A = \angle B$
- (ii) $\angle C = \angle D$
- (iii) $\triangle ABC \cong \triangle BAD$
- (iv) diagonal $AC =$ diagonal BD

[Hint: Extend AB and draw a line through C parallel to DA intersecting AB produced at E.]

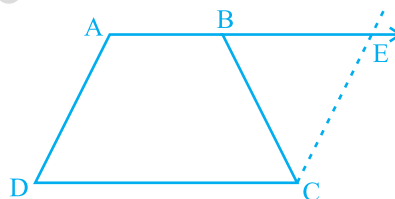


Fig. 8.14

8.2 The Mid-point Theorem

You have studied many properties of a triangle as well as a quadrilateral. Now let us study yet another result which is related to the mid-point of sides of a triangle. Perform the following activity.

Draw a triangle and mark the mid-points E and F of two sides of the triangle. Join the points E and F (see Fig. 8.15).

Measure EF and BC. Measure $\angle AEF$ and $\angle ABC$.

What do you observe? You will find that :

$$EF = \frac{1}{2} BC \text{ and } \angle AEF = \angle ABC$$

so, $EF \parallel BC$

Repeat this activity with some more triangles.

So, you arrive at the following theorem:

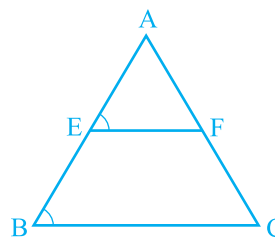


Fig. 8.15

Theorem 8.8 : *The line segment joining the mid-points of two sides of a triangle is parallel to the third side.*

You can prove this theorem using the following clue:

Observe Fig 8.16 in which E and F are mid-points of AB and AC respectively and $CD \parallel BA$.

$$\triangle AEF \cong \triangle CDF \quad (\text{ASA Rule})$$

So, $EF = DF$ and $BE = AE = DC$ (Why?)

Therefore, BCDE is a parallelogram. (Why?)

This gives $EF \parallel BC$.

In this case, also note that $EF = \frac{1}{2} ED = \frac{1}{2} BC$.

Can you state the converse of Theorem 8.8? Is the converse true?

You will see that converse of the above theorem is also true which is stated as below:

Theorem 8.9 : *The line drawn through the mid-point of one side of a triangle, parallel to another side bisects the third side.*

In Fig 8.17, observe that E is the mid-point of AB, line l is passing through E and is parallel to BC and $CM \parallel BA$.

Prove that $AF = CF$ by using the congruence of $\triangle AEF$ and $\triangle CDF$.

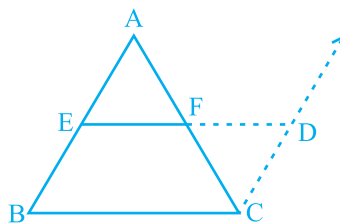


Fig. 8.16

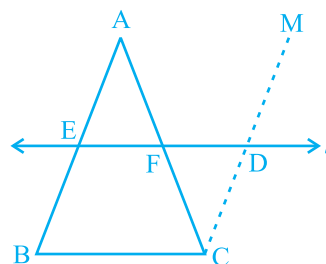


Fig. 8.17

Example 6 : In $\triangle ABC$, D, E and F are respectively the mid-points of sides AB, BC and CA (see Fig. 8.18). Show that $\triangle ABC$ is divided into four congruent triangles by joining D, E and F.

Solution : As D and E are mid-points of sides AB and BC of the triangle ABC, by Theorem 8.8,

$$DE \parallel AC$$

Similarly, $DF \parallel BC$ and $EF \parallel AB$

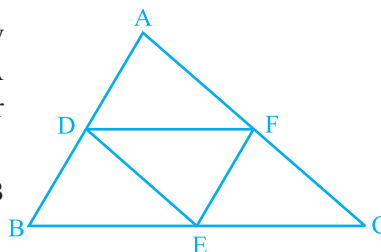


Fig. 8.18

Therefore ADEF, BDFE and DFCE are all parallelograms.

Now DE is a diagonal of the parallelogram BDFE,

therefore, $\triangle BDE \cong \triangle FED$

Similarly $\triangle DAF \cong \triangle FED$

and $\triangle EFC \cong \triangle FED$

So, all the four triangles are congruent.

Example 7 : l , m and n are three parallel lines intersected by transversals p and q such that l , m and n cut off equal intercepts AB and BC on p (see Fig. 8.19). Show that l , m and n cut off equal intercepts DE and EF on q also.

Solution : We are given that $AB = BC$ and have to prove that $DE = EF$.

Let us join A to F intersecting m at G.

The trapezium ACFD is divided into two triangles; namely $\triangle ACF$ and $\triangle AFD$.

In $\triangle ACF$, it is given that B is the mid-point of AC ($AB = BC$)

and $BG \parallel CF$ (since $m \parallel n$).

So, G is the mid-point of AF (by using Theorem 8.9)

Now, in $\triangle AFD$, we can apply the same argument as G is the mid-point of AF, $GE \parallel AD$ and so by Theorem 8.9, E is the mid-point of DF,

i.e., $DE = EF$.

In other words, l , m and n cut off equal intercepts on q also.

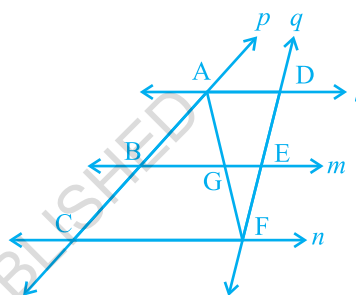


Fig. 8.19

EXERCISE 8.2

1. ABCD is a quadrilateral in which P, Q, R and S are mid-points of the sides AB, BC, CD and DA (see Fig 8.20). AC is a diagonal. Show that :
 - (i) $SR \parallel AC$ and $SR = \frac{1}{2} AC$
 - (ii) $PQ = SR$
 - (iii) PQRS is a parallelogram.

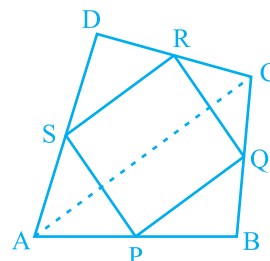


Fig. 8.20

2. ABCD is a rhombus and P, Q, R and S are the mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral PQRS is a rectangle.
3. ABCD is a rectangle and P, Q, R and S are mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral PQRS is a rhombus.
4. ABCD is a trapezium in which $AB \parallel DC$, BD is a diagonal and E is the mid-point of AD. A line is drawn through E parallel to AB intersecting BC at F (see Fig. 8.21). Show that F is the mid-point of BC.

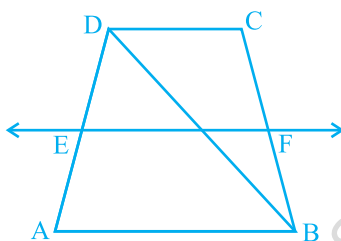


Fig. 8.21

5. In a parallelogram ABCD, E and F are the mid-points of sides AB and CD respectively (see Fig. 8.22). Show that the line segments AF and EC trisect the diagonal BD.

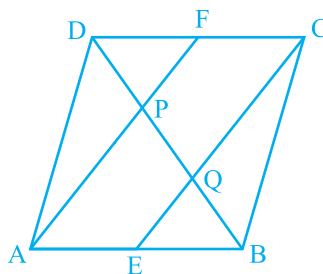


Fig. 8.22

6. ABC is a triangle right angled at C. A line through the mid-point M of hypotenuse AB and parallel to BC intersects AC at D. Show that
 - (i) D is the mid-point of AC
 - (ii) $MD \perp AC$
 - (iii) $CM = MA = \frac{1}{2} AB$

8.3 Summary

In this chapter, you have studied the following points :

1. A diagonal of a parallelogram divides it into two congruent triangles.
2. In a parallelogram,
 - (i) opposite sides are equal
 - (ii) opposite angles are equal
 - (iii) diagonals bisect each other
3. Diagonals of a rectangle bisect each other and are equal and vice-versa.
4. Diagonals of a rhombus bisect each other at right angles and vice-versa.
5. Diagonals of a square bisect each other at right angles and are equal, and vice-versa.
6. The line-segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.
7. A line through the mid-point of a side of a triangle parallel to another side bisects the third side.

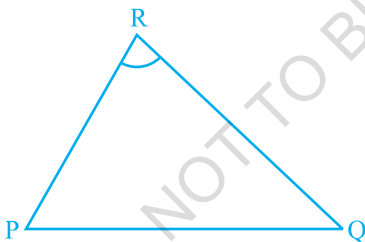
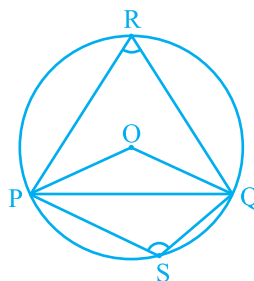


0962CH10

CHAPTER 9**CIRCLES****9.1 Angle Subtended by a Chord at a Point**

You have already studied about circles and its parts in Class VI.

Take a line segment PQ and a point R not on the line containing PQ. Join PR and QR (see Fig. 9.1). Then $\angle PRQ$ is called the angle subtended by the line segment PQ at the point R. What are angles POQ, PRQ and PSQ called in Fig. 9.2? $\angle POQ$ is the angle subtended by the chord PQ at the centre O, $\angle PRQ$ and $\angle PSQ$ are respectively the angles subtended by PQ at points R and S on the major and minor arcs PQ.

**Fig. 9.1****Fig. 9.2**

Let us examine the relationship between the size of the chord and the angle subtended by it at the centre. You may see by drawing different chords of a circle and angles subtended by them at the centre that the longer is the chord, the bigger will be the angle subtended by it at the centre. What will happen if you take two equal chords of a circle? Will the angles subtended at the centre be the same or not?

Draw two or more equal chords of a circle and measure the angles subtended by them at the centre (see Fig.9.3). You will find that the angles subtended by them at the centre are equal. Let us give a proof of this fact.

Theorem 9.1 : *Equal chords of a circle subtend equal angles at the centre.*

Proof : You are given two equal chords AB and CD of a circle with centre O (see Fig.9.4). You want to prove that $\angle AOB = \angle COD$.

In triangles AOB and COD,

$$OA = OC \quad (\text{Radii of a circle})$$

$$OB = OD \quad (\text{Radii of a circle})$$

$$AB = CD \quad (\text{Given})$$

$$\text{Therefore, } \triangle AOB \cong \triangle COD \quad (\text{SSS rule})$$

$$\text{This gives } \angle AOB = \angle COD$$

(Corresponding parts of congruent triangles)

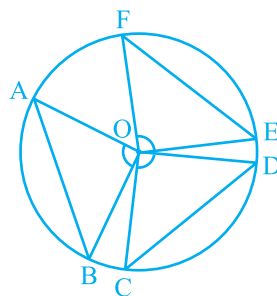


Fig. 9.3

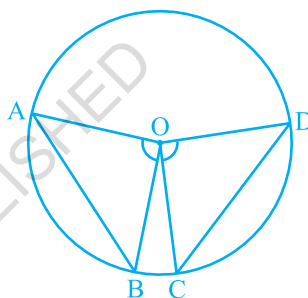


Fig. 9.4

Remark : For convenience, the abbreviation CPCT will be used in place of 'Corresponding parts of congruent triangles', because we use this very frequently as you will see.

Now if two chords of a circle subtend equal angles at the centre, what can you say about the chords? Are they equal or not? Let us examine this by the following activity:

Take a tracing paper and trace a circle on it. Cut it along the circle to get a disc. At its centre O, draw an angle AOB where A, B are points on the circle. Make another angle POQ at the centre equal to $\angle AOB$. Cut the disc along AB and PQ (see Fig. 9.5). You will get two segments ACB and PRQ of the circle. If you put one on the other, what do you observe? They cover each other, i.e., they are congruent. So $AB = PQ$.

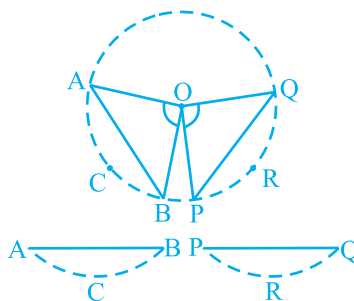


Fig. 9.5

Though you have seen it for this particular case, try it out for other equal angles too. The chords will all turn out to be equal because of the following theorem:

Theorem 9.2 : *If the angles subtended by the chords of a circle at the centre are equal, then the chords are equal.*

The above theorem is the converse of the Theorem 9.1. Note that in Fig. 9.4, if you take $\angle AOB = \angle COD$, then

$$\triangle AOB \cong \triangle COD \text{ (Why?)}$$

Can you now see that $AB = CD$?

EXERCISE 9.1

1. Recall that two circles are congruent if they have the same radii. Prove that equal chords of congruent circles subtend equal angles at their centres.
2. Prove that if chords of congruent circles subtend equal angles at their centres, then the chords are equal.

9.2 Perpendicular from the Centre to a Chord

Activity : Draw a circle on a tracing paper. Let O be its centre. Draw a chord AB. Fold the paper along a line through O so that a portion of the chord falls on the other. Let the crease cut AB at the point M. Then, $\angle OMA = \angle OMB = 90^\circ$ or OM is perpendicular to AB. Does the point B coincide with A (see Fig.9.6)?

Yes it will. So $MA = MB$.

Give a proof yourself by joining OA and OB and proving the right triangles OMA and OMB to be congruent. This example is a particular instance of the following result:

Theorem 9.3 : *The perpendicular from the centre of a circle to a chord bisects the chord.*

What is the converse of this theorem? To write this, first let us be clear what is assumed in Theorem 9.3 and what is proved. Given that the perpendicular from the centre of a circle to a chord is drawn and to prove that it bisects the chord. Thus in the converse, what the hypothesis is 'if a line from the centre bisects a chord of a circle' and what is to be proved is 'the line is perpendicular to the chord'. So the converse is:

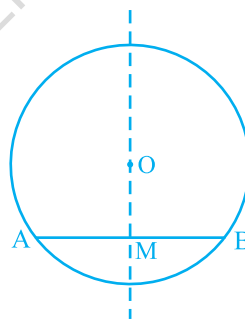


Fig. 9.6

Theorem 9.4 : *The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.*

Is this true? Try it for few cases and see. You will see that it is true for these cases. See if it is true, in general, by doing the following exercise. We will write the stages and you give the reasons.

Let AB be a chord of a circle with centre O and O is joined to the mid-point M of AB. You have to prove that $OM \perp AB$. Join OA and OB (see Fig. 9.7). In triangles OAM and OBM,

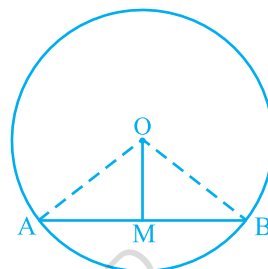


Fig. 9.7

$$OA = OB \quad (\text{Why ?})$$

$$AM = BM \quad (\text{Why ?})$$

$$OM = OM \quad (\text{Common})$$

$$\text{Therefore, } \triangle OAM \cong \triangle OBM \quad (\text{How ?})$$

$$\text{This gives } \angle OMA = \angle OMB = 90^\circ \quad (\text{Why ?})$$

9.3 Equal Chords and their Distances from the Centre

Let AB be a line and P be a point. Since there are infinite numbers of points on a line, if you join these points to P, you will get infinitely many line segments $PL_1, PL_2, PM, PL_3, PL_4$, etc. Which of these is the distance of AB from P? You may think a while and get the answer. Out of these line segments, the perpendicular from P to AB, namely PM in Fig. 9.8, will be the least. In Mathematics, we define this least length PM to be **the distance of AB from P**. So you may say that:

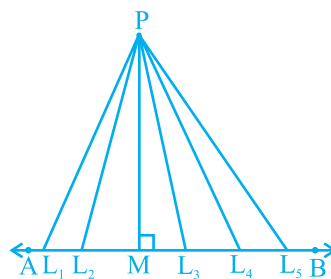


Fig. 9.8

The length of the perpendicular from a point to a line is the distance of the line from the point.

Note that if the point lies on the line, the distance of the line from the point is zero.

A circle can have infinitely many chords. You may observe by drawing chords of a circle that longer chord is nearer to the centre than the smaller chord. You may observe it by drawing several chords of a circle of different lengths and measuring their distances from the centre. What is the distance of the diameter, which is the

longest chord from the centre? Since the centre lies on it, the distance is zero. Do you think that there is some relationship between the length of chords and their distances from the centre? Let us see if this is so.

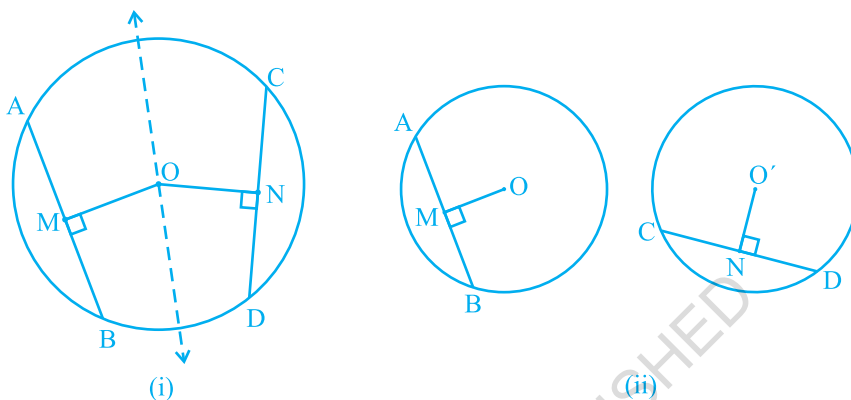


Fig. 9.9

Activity : Draw a circle of any radius on a tracing paper. Draw two equal chords AB and CD of it and also the perpendiculars OM and ON on them from the centre O. Fold the figure so that D falls on B and C falls on A [see Fig. 9.9 (i)]. You may observe that O lies on the crease and N falls on M. Therefore, $OM = ON$. Repeat the activity by drawing congruent circles with centres O and O' and taking equal chords AB and CD one on each. Draw perpendiculars OM and $O'N$ on them [see Fig. 9.9(ii)]. Cut one circular disc and put it on the other so that AB coincides with CD. Then you will find that O coincides with O' and M coincides with N. In this way you verified the following:

Theorem 9.5 : *Equal chords of a circle (or of congruent circles) are equidistant from the centre (or centres).*

Next, it will be seen whether the converse of this theorem is true or not. For this, draw a circle with centre O. From the centre O, draw two line segments OL and OM of equal length and lying inside the circle [see Fig. 9.10(i)]. Then draw chords PQ and RS of the circle perpendicular to OL and OM respectively [see Fig. 9.10(ii)]. Measure the lengths of PQ and RS. Are these different? No, both are equal. Repeat the activity for more equal line segments and drawing the chords perpendicular to them. This verifies the converse of the Theorem 9.5 which is stated as follows:

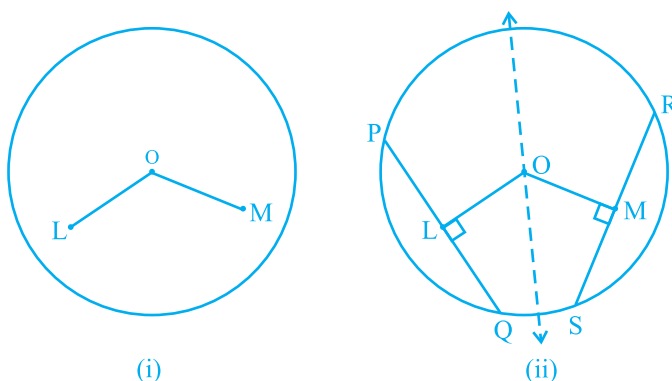


Fig. 9.10

Theorem 9.6 : *Chords equidistant from the centre of a circle are equal in length.*

We now take an example to illustrate the use of the above results:

Example 1 : If two intersecting chords of a circle make equal angles with the diameter passing through their point of intersection, prove that the chords are equal.

Solution : Given that AB and CD are two chords of a circle, with centre O intersecting at a point E. PQ is a diameter through E, such that $\angle AEQ = \angle DEQ$ (see Fig.9.11). You have to prove that $AB = CD$. Draw perpendiculars OL and OM on chords AB and CD, respectively. Now

$$\begin{aligned}\angle LOE &= 180^\circ - 90^\circ - \angle LEO = 90^\circ - \angle LEO \\ &\quad \text{(Angle sum property of a triangle)} \\ &= 90^\circ - \angle AEQ = 90^\circ - \angle DEQ \\ &= 90^\circ - \angle MEO = \angle MOE\end{aligned}$$

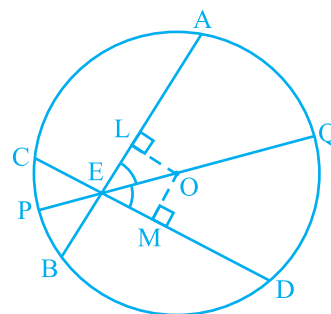


Fig. 9.11

In triangles OLE and OME,

$$\angle LEO = \angle MEO \quad (\text{Why ?})$$

$$\angle LOE = \angle MOE \quad (\text{Proved above})$$

$$EO = EO \quad (\text{Common})$$

$$\text{Therefore, } \triangle OLE \cong \triangle OME \quad (\text{Why ?})$$

$$\text{This gives } OL = OM \quad (\text{CPCT})$$

$$\text{So, } AB = CD \quad (\text{Why ?})$$

EXERCISE 9.2

- Two circles of radii 5 cm and 3 cm intersect at two points and the distance between their centres is 4 cm. Find the length of the common chord.
- If two equal chords of a circle intersect within the circle, prove that the segments of one chord are equal to corresponding segments of the other chord.
- If two equal chords of a circle intersect within the circle, prove that the line joining the point of intersection to the centre makes equal angles with the chords.
- If a line intersects two concentric circles (circles with the same centre) with centre O at A, B, C and D, prove that $AB = CD$ (see Fig. 9.12).
- Three girls Reshma, Salma and Mandip are playing a game by standing on a circle of radius 5m drawn in a park. Reshma throws a ball to Salma, Salma to Mandip, Mandip to Reshma. If the distance between Reshma and Salma and between Salma and Mandip is 6m each, what is the distance between Reshma and Mandip?
- A circular park of radius 20m is situated in a colony. Three boys Ankur, Syed and David are sitting at equal distance on its boundary each having a toy telephone in his hands to talk each other. Find the length of the string of each phone.

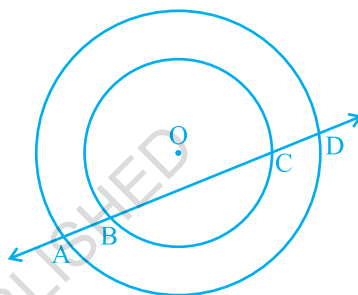


Fig. 9.12

9.4 Angle Subtended by an Arc of a Circle

You have seen that the end points of a chord other than diameter of a circle cuts it into two arcs – one major and other minor. If you take two equal chords, what can you say about the size of arcs? Is one arc made by first chord equal to the corresponding arc made by another chord? In fact, they are more than just equal in length. They are congruent in the sense that if one arc is put on the other, without bending or twisting, one superimposes the other completely.

You can verify this fact by cutting the arc, corresponding to the chord CD from the circle along CD and put it on the corresponding arc made by equal chord AB. You will find that the arc CD superimpose the arc AB completely (see Fig. 9.13). This shows that equal chords make congruent arcs and conversely congruent arcs make equal chords of a circle. You can state it as follows:

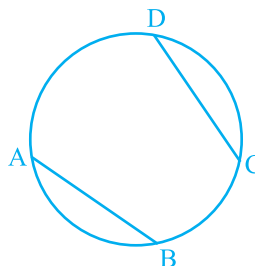


Fig. 9.13

If two chords of a circle are equal, then their corresponding arcs are congruent and conversely, if two arcs are congruent, then their corresponding chords are equal.

Also the angle subtended by an arc at the centre is defined to be angle subtended by the corresponding chord at the centre in the sense that the minor arc subtends the angle and the major arc subtends the reflex angle. Therefore, in Fig 9.14, the angle subtended by the minor arc PQ at O is $\angle POQ$ and the angle subtended by the major arc PQ at O is reflex angle POQ.

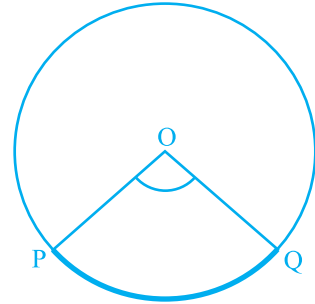


Fig. 9.14

In view of the property above and Theorem 9.1, the following result is true:

Congruent arcs (or equal arcs) of a circle subtend equal angles at the centre.

Therefore, the angle subtended by a chord of a circle at its centre is equal to the angle subtended by the corresponding (minor) arc at the centre. The following theorem gives the relationship between the angles subtended by an arc at the centre and at a point on the circle.

Theorem 9.7 : *The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.*

Proof : Given an arc PQ of a circle subtending angles POQ at the centre O and PAQ at a point A on the remaining part of the circle. We need to prove that $\angle POQ = 2 \angle PAQ$.

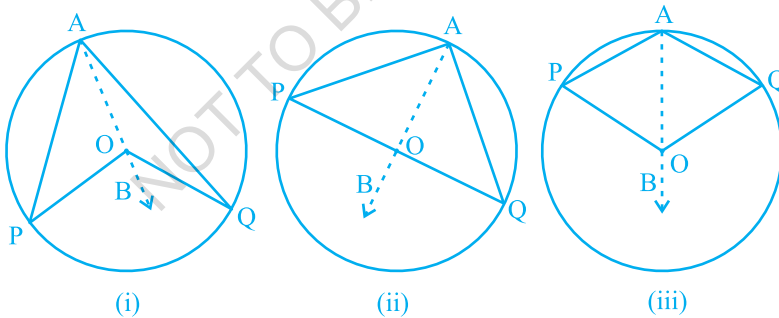


Fig. 9.15

Consider the three different cases as given in Fig. 9.15. In (i), arc PQ is minor; in (ii), arc PQ is a semicircle and in (iii), arc PQ is major.

Let us begin by joining AO and extending it to a point B.

In all the cases,

$$\angle BOQ = \angle OAQ + \angle AQO$$

because an exterior angle of a triangle is equal to the sum of the two interior opposite angles.

Also in $\triangle OAQ$,

$$OA = OQ \quad (\text{Radii of a circle})$$

$$\text{Therefore, } \angle OAQ = \angle OQA \quad (\text{Theorem 7.5})$$

$$\text{This gives } \angle BOQ = 2 \angle OAQ \quad (1)$$

$$\text{Similarly, } \angle BOP = 2 \angle OAP \quad (2)$$

$$\text{From (1) and (2), } \angle BOP + \angle BOQ = 2(\angle OAP + \angle OAQ)$$

$$\text{This is the same as } \angle POQ = 2 \angle PAQ \quad (3)$$

For the case (iii), where PQ is the major arc, (3) is replaced by

$$\text{reflex angle } POQ = 2 \angle PAQ$$

Remark : Suppose we join points P and Q and form a chord PQ in the above figures. Then $\angle PAQ$ is also called the angle formed in the segment PAQP.

In Theorem 9.7, A can be any point on the remaining part of the circle. So if you take any other point C on the remaining part of the circle (see Fig. 9.16), you have

$$\angle POQ = 2 \angle PCQ = 2 \angle PAQ$$

$$\text{Therefore, } \angle PCQ = \angle PAQ.$$

This proves the following:

Theorem 9.8 : *Angles in the same segment of a circle are equal.*

Again let us discuss the case (ii) of Theorem 9.8 separately. Here $\angle PAQ$ is an angle in the segment, which is a semicircle. Also, $\angle PAQ = \frac{1}{2} \angle POQ = \frac{1}{2} \times 180^\circ = 90^\circ$. If you take any other point C on the semicircle, again you get that

$$\angle PCQ = 90^\circ$$

Therefore, you find another property of the circle as:

Angle in a semicircle is a right angle.

The converse of Theorem 9.8 is also true. It can be stated as:

Theorem 9.9 : *If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle (i.e. they are concyclic).*

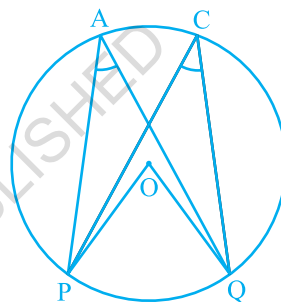


Fig. 9.16

You can see the truth of this result as follows:

In Fig. 9.17, AB is a line segment, which subtends equal angles at two points C and D. That is

$$\angle ACB = \angle ADB$$

To show that the points A, B, C and D lie on a circle let us draw a circle through the points A, C and B. Suppose it does not pass through the point D. Then it will intersect AD (or extended AD) at a point, say E (or E').

If points A, C, E and B lie on a circle,

$$\angle ACB = \angle AEB \quad (\text{Why?})$$

But it is given that $\angle ACB = \angle ADB$.

Therefore, $\angle AEB = \angle ADB$.

This is not possible unless E coincides with D. (Why?)

Similarly, E' should also coincide with D.

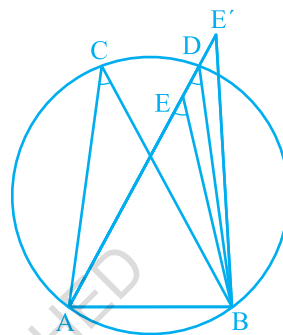


Fig. 9.17

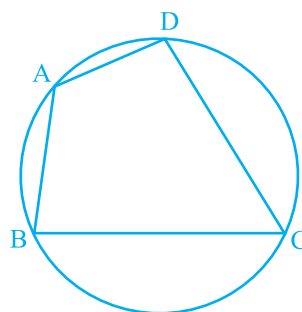


Fig. 9.18

9.5 Cyclic Quadrilaterals

A quadrilateral ABCD is called *cyclic* if all the four vertices of it lie on a circle (see Fig 9.18). You will find a peculiar property in such quadrilaterals. Draw several cyclic quadrilaterals of different sides and name each of these as ABCD. (This can be done by drawing several circles of different radii and taking four points on each of them.) Measure the opposite angles and write your observations in the following table.

S.No. of Quadrilateral	$\angle A$	$\angle B$	$\angle C$	$\angle D$	$\angle A + \angle C$	$\angle B + \angle D$
1.						
2.						
3.						
4.						
5.						
6.						

What do you infer from the table?

You find that $\angle A + \angle C = 180^\circ$ and $\angle B + \angle D = 180^\circ$, neglecting the error in measurements. This verifies the following:

Theorem 9.10 : *The sum of either pair of opposite angles of a cyclic quadrilateral is 180° .*

In fact, the converse of this theorem, which is stated below is also true.

Theorem 9.11 : *If the sum of a pair of opposite angles of a quadrilateral is 180° , the quadrilateral is cyclic.*

You can see the truth of this theorem by following a method similar to the method adopted for Theorem 9.9.

Example 2 : In Fig. 9.19, AB is a diameter of the circle, CD is a chord equal to the radius of the circle. AC and BD when extended intersect at a point E. Prove that $\angle AEB = 60^\circ$.

Solution : Join OC, OD and BC.

Triangle ODC is equilateral (Why?)

Therefore, $\angle COD = 60^\circ$

Now, $\angle CBD = \frac{1}{2} \angle COD$ (Theorem 9.7)

This gives $\angle CBD = 30^\circ$

Again, $\angle ACB = 90^\circ$ (Why ?)

So, $\angle BCE = 180^\circ - \angle ACB = 90^\circ$

Which gives $\angle CEB = 90^\circ - 30^\circ = 60^\circ$, i.e., $\angle AEB = 60^\circ$

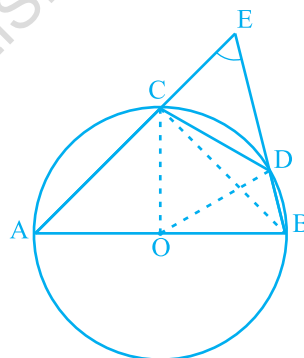


Fig. 9.19

Example 3 : In Fig 9.20, ABCD is a cyclic quadrilateral in which AC and BD are its diagonals. If $\angle DBC = 55^\circ$ and $\angle BAC = 45^\circ$, find $\angle BCD$.

Solution : $\angle CAD = \angle DBC = 55^\circ$
(Angles in the same segment)

Therefore, $\angle DAB = \angle CAD + \angle BAC$
 $= 55^\circ + 45^\circ = 100^\circ$

But $\angle DAB + \angle BCD = 180^\circ$

(Opposite angles of a cyclic quadrilateral)

So, $\angle BCD = 180^\circ - 100^\circ = 80^\circ$

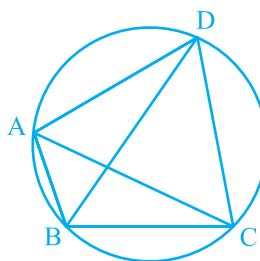


Fig. 9.20

Example 4 : Two circles intersect at two points A and B. AD and AC are diameters to the two circles (see Fig. 9.21). Prove that B lies on the line segment DC.

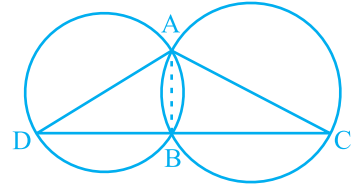


Fig. 9.21

Solution : Join AB.

$$\angle ABD = 90^\circ \quad (\text{Angle in a semicircle})$$

$$\angle ABC = 90^\circ \quad (\text{Angle in a semicircle})$$

$$\text{So, } \angle ABD + \angle ABC = 90^\circ + 90^\circ = 180^\circ$$

Therefore, DBC is a line. That is B lies on the line segment DC.

Example 5 : Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadrilateral is cyclic.

Solution : In Fig. 9.22, ABCD is a quadrilateral in which the angle bisectors AH, BF, CF and DH of internal angles A, B, C and D respectively form a quadrilateral EFGH.

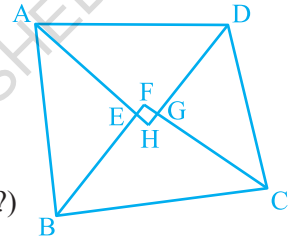


Fig. 9.22

$$\text{Now, } \angle FEH = \angle AEB = 180^\circ - \angle EAB - \angle EBA \quad (\text{Why ?})$$

$$= 180^\circ - \frac{1}{2} (\angle A + \angle B)$$

$$\text{and } \angle FGH = \angle CGD = 180^\circ - \angle GCD - \angle GDC \quad (\text{Why ?})$$

$$= 180^\circ - \frac{1}{2} (\angle C + \angle D)$$

$$\text{Therefore, } \angle FEH + \angle FGH = 180^\circ - \frac{1}{2} (\angle A + \angle B) + 180^\circ - \frac{1}{2} (\angle C + \angle D)$$

$$= 360^\circ - \frac{1}{2} (\angle A + \angle B + \angle C + \angle D) = 360^\circ - \frac{1}{2} \times 360^\circ$$

$$= 360^\circ - 180^\circ = 180^\circ$$

Therefore, by Theorem 9.11, the quadrilateral EFGH is cyclic.

EXERCISE 9.3

1. In Fig. 9.23, A, B and C are three points on a circle with centre O such that $\angle BOC = 30^\circ$ and $\angle AOB = 60^\circ$. If D is a point on the circle other than the arc ABC, find $\angle ADC$.

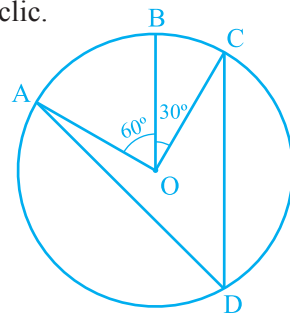


Fig. 9.23

2. A chord of a circle is equal to the radius of the circle. Find the angle subtended by the chord at a point on the minor arc and also at a point on the major arc.
3. In Fig. 9.24, $\angle PQR = 100^\circ$, where P, Q and R are points on a circle with centre O. Find $\angle OPR$.

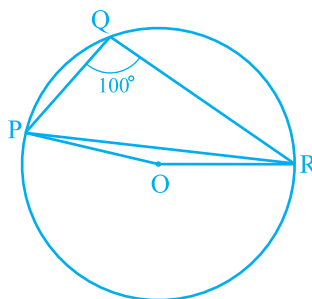


Fig. 9.24

4. In Fig. 9.25, $\angle ABC = 69^\circ$, $\angle ACB = 31^\circ$, find $\angle BDC$.

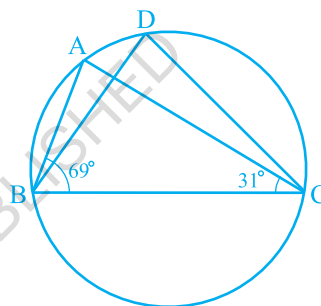


Fig. 9.25

5. In Fig. 9.26, A, B, C and D are four points on a circle. AC and BD intersect at a point E such that $\angle BEC = 130^\circ$ and $\angle ECD = 20^\circ$. Find $\angle BAC$.

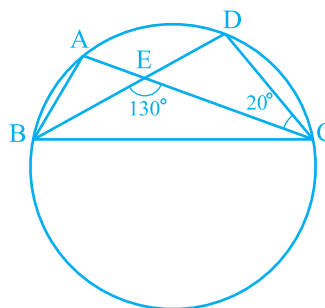


Fig. 9.26

6. ABCD is a cyclic quadrilateral whose diagonals intersect at a point E. If $\angle DBC = 70^\circ$, $\angle BAC$ is 30° , find $\angle BCD$. Further, if $AB = BC$, find $\angle ECD$.
7. If diagonals of a cyclic quadrilateral are diameters of the circle through the vertices of the quadrilateral, prove that it is a rectangle.
8. If the non-parallel sides of a trapezium are equal, prove that it is cyclic.

9. Two circles intersect at two points B and C. Through B, two line segments ABD and PBQ are drawn to intersect the circles at A, D and P, Q respectively (see Fig. 9.27). Prove that $\angle ACP = \angle QCD$.

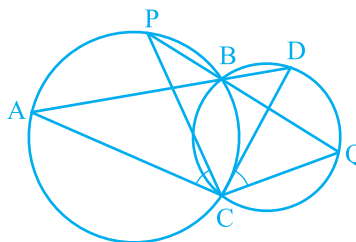


Fig. 9.27

10. If circles are drawn taking two sides of a triangle as diameters, prove that the point of intersection of these circles lie on the third side.
11. ABC and ADC are two right triangles with common hypotenuse AC. Prove that $\angle CAD = \angle CBD$.
12. Prove that a cyclic parallelogram is a rectangle.

9.6 Summary

In this chapter, you have studied the following points:

1. A circle is the collection of all points in a plane, which are equidistant from a fixed point in the plane.
2. Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.
3. If the angles subtended by two chords of a circle (or of congruent circles) at the centre (corresponding centres) are equal, the chords are equal.
4. The perpendicular from the centre of a circle to a chord bisects the chord.
5. The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.
6. Equal chords of a circle (or of congruent circles) are equidistant from the centre (or corresponding centres).
7. Chords equidistant from the centre (or corresponding centres) of a circle (or of congruent circles) are equal.
8. If two arcs of a circle are congruent, then their corresponding chords are equal and conversely if two chords of a circle are equal, then their corresponding arcs (minor, major) are congruent.
9. Congruent arcs of a circle subtend equal angles at the centre.
10. The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.
11. Angles in the same segment of a circle are equal.

12. Angle in a semicircle is a right angle.
13. If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle.
14. The sum of either pair of opposite angles of a cyclic quadrilateral is 180° .
15. If sum of a pair of opposite angles of a quadrilateral is 180° , the quadrilateral is cyclic.

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CHAPTER 10

HERON'S FORMULA

10.1 Area of a Triangle — by Heron's Formula

We know that the area of triangle when its height is given, is $\frac{1}{2} \times \text{base} \times \text{height}$. Now suppose that we know the lengths of the sides of a scalene triangle and not the height. Can you still find its area? For instance, you have a triangular park whose sides are 40 m, 32 m, and 24 m. How will you calculate its area? Definitely if you want to apply the formula, you will have to calculate its height. But we do not have a clue to calculate the height. Try doing so. If you are not able to get it, then go to the next section.

Heron was born in about 10AD possibly in Alexandria in Egypt. He worked in applied mathematics. His works on mathematical and physical subjects are so numerous and varied that he is considered to be an encyclopedic writer in these fields. His geometrical works deal largely with problems on mensuration written in three books. Book I deals with the area of squares, rectangles, triangles, trapezoids (trapezia), various other specialised quadrilaterals, the regular polygons, circles, surfaces of cylinders, cones, spheres etc. In this book, Heron has derived the famous formula for the area of a triangle in terms of its three sides.



Heron (10 C.E. – 75 C.E.)

Fig. 10.1

The formula given by Heron about the area of a triangle, is also known as *Heron's formula*. It is stated as:

$$\text{Area of a triangle} = \sqrt{s(s-a)(s-b)(s-c)} \quad (1)$$

where a , b and c are the sides of the triangle, and s = semi-perimeter, i.e., half the perimeter of the triangle = $\frac{a + b + c}{2}$,

This formula is helpful where it is not possible to find the height of the triangle easily. Let us apply it to calculate the area of the triangular park ABC, mentioned above (see Fig. 10.2).

Let us take $a = 40$ m, $b = 24$ m, $c = 32$ m,

so that we have $s = \frac{40 + 24 + 32}{2}$ m = 48 m.

$$s - a = (48 - 40) \text{ m} = 8 \text{ m},$$

$$s - b = (48 - 24) \text{ m} = 24 \text{ m},$$

$$s - c = (48 - 32) \text{ m} = 16 \text{ m}.$$

Therefore, area of the park ABC

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{48 \times 8 \times 24 \times 16} \text{ m}^2 = 384 \text{ m}^2$$

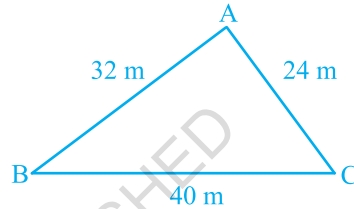


Fig. 10.2

We see that $32^2 + 24^2 = 1024 + 576 = 1600 = 40^2$. Therefore, the sides of the park make a right triangle. The largest side, i.e., BC which is 40 m will be the hypotenuse and the angle between the sides AB and AC will be 90° .

We can check that the area of the park is $\frac{1}{2} \times 32 \times 24 \text{ m}^2 = 384 \text{ m}^2$.

We find that the area we have got is the same as we found by using Heron's formula.

Now using Heron's formula, you verify this fact by finding the areas of other triangles discussed earlier viz.,

- (i) equilateral triangle with side 10 cm.
- (ii) isosceles triangle with unequal side as 8 cm and each equal side as 5 cm.

You will see that

For (i), we have $s = \frac{10 + 10 + 10}{2}$ cm = 15 cm.

$$\begin{aligned}\text{Area of triangle} &= \sqrt{15(15-10)(15-10)(15-10)} \text{ cm}^2 \\ &= \sqrt{15 \times 5 \times 5 \times 5} \text{ cm}^2 = 25\sqrt{3} \text{ cm}^2\end{aligned}$$

For (ii), we have $s = \frac{8+5+5}{2} \text{ cm} = 9 \text{ cm}$

$$\text{Area of triangle} = \sqrt{9(9-8)(9-5)(9-5)} \text{ cm}^2 = \sqrt{9 \times 1 \times 4 \times 4} \text{ cm}^2 = 12 \text{ cm}^2.$$

Let us now solve some more examples:

Example 1 : Find the area of a triangle, two sides of which are 8 cm and 11 cm and the perimeter is 32 cm (see Fig. 10.3).

Solution : Here we have perimeter of the triangle = 32 cm, $a = 8$ cm and $b = 11$ cm.

$$\text{Third side } c = 32 \text{ cm} - (8 + 11) \text{ cm} = 13 \text{ cm}$$

$$\text{So, } 2s = 32, \text{ i.e., } s = 16 \text{ cm,}$$

$$s - a = (16 - 8) \text{ cm} = 8 \text{ cm,}$$

$$s - b = (16 - 11) \text{ cm} = 5 \text{ cm,}$$

$$s - c = (16 - 13) \text{ cm} = 3 \text{ cm.}$$

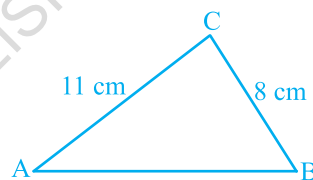


Fig. 10.3

$$\begin{aligned}\text{Therefore, area of the triangle} &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{16 \times 8 \times 5 \times 3} \text{ cm}^2 = 8\sqrt{30} \text{ cm}^2\end{aligned}$$

Example 2 : A triangular park ABC has sides 120m, 80m and 50m (see Fig. 10.4). A gardener *Dhania* has to put a fence all around it and also plant grass inside. How much area does she need to plant? Find the cost of fencing it with barbed wire at the rate of ₹20 per metre leaving a space 3m wide for a gate on one side.

Solution : For finding area of the park, we have

$$2s = 50 \text{ m} + 80 \text{ m} + 120 \text{ m} = 250 \text{ m.}$$

$$\text{i.e., } s = 125 \text{ m}$$

$$\text{Now, } s - a = (125 - 120) \text{ m} = 5 \text{ m,}$$

$$s - b = (125 - 80) \text{ m} = 45 \text{ m,}$$

$$s - c = (125 - 50) \text{ m} = 75 \text{ m.}$$

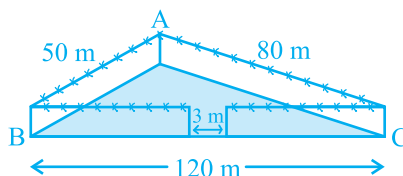


Fig. 10.4

$$\begin{aligned}
 \text{Therefore, area of the park} &= \sqrt{s(s-a)(s-b)(s-c)} \\
 &= \sqrt{125 \times 5 \times 45 \times 75} \text{ m}^2 \\
 &= 375\sqrt{15} \text{ m}^2
 \end{aligned}$$

Also, perimeter of the park = $AB + BC + CA = 250 \text{ m}$

Therefore, length of the wire needed for fencing = $250 \text{ m} - 3 \text{ m}$ (to be left for gate)
 $= 247 \text{ m}$

And so the cost of fencing = $\text{₹}20 \times 247 = \text{₹}4940$

Example 3 : The sides of a triangular plot are in the ratio of 3 : 5 : 7 and its perimeter is 300 m. Find its area.

Solution : Suppose that the sides, in metres, are $3x$, $5x$ and $7x$ (see Fig. 10.5).

Then, we know that $3x + 5x + 7x = 300$ (perimeter of the triangle)

Therefore, $15x = 300$, which gives $x = 20$.

So the sides of the triangle are $3 \times 20 \text{ m}$, $5 \times 20 \text{ m}$ and $7 \times 20 \text{ m}$

i.e., 60 m, 100 m and 140 m.

Can you now find the area [Using Heron's formula]?

$$\text{We have } s = \frac{60 + 100 + 140}{2} \text{ m} = 150 \text{ m},$$

$$\begin{aligned}
 \text{and area will be } &\sqrt{150(150-60)(150-100)(150-140)} \text{ m}^2 \\
 &= \sqrt{150 \times 90 \times 50 \times 10} \text{ m}^2 \\
 &= 1500\sqrt{3} \text{ m}^2
 \end{aligned}$$

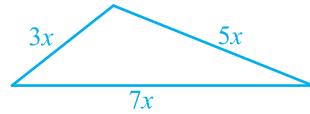


Fig. 10.5

EXERCISE 10.1

1. A traffic signal board, indicating 'SCHOOL AHEAD', is an equilateral triangle with side ' a '. Find the area of the signal board, using Heron's formula. If its perimeter is 180 cm, what will be the area of the signal board?

2. The triangular side walls of a flyover have been used for advertisements. The sides of the walls are 122 m, 22 m and 120 m (see Fig. 10.6). The advertisements yield an earning of ₹ 5000 per m^2 per year. A company hired one of its walls for 3 months. How much rent did it pay?

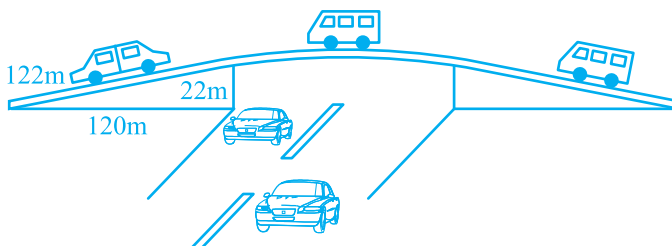


Fig. 10.6

3. There is a slide in a park. One of its side walls has been painted in some colour with a message “KEEP THE PARK GREEN AND CLEAN” (see Fig. 10.7). If the sides of the wall are 15 m, 11 m and 6 m, find the area painted in colour.

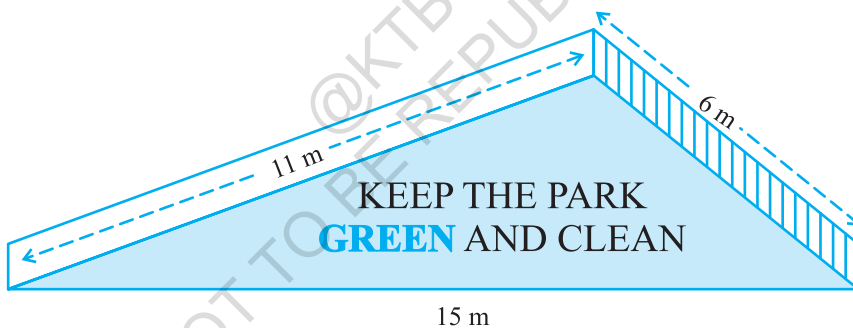


Fig. 10.7

4. Find the area of a triangle two sides of which are 18cm and 10cm and the perimeter is 42cm.
5. Sides of a triangle are in the ratio of 12 : 17 : 25 and its perimeter is 540cm. Find its area.
6. An isosceles triangle has perimeter 30 cm and each of the equal sides is 12 cm. Find the area of the triangle.

10.2 Summary

In this chapter, you have studied the following points :

1. Area of a triangle with its sides as a , b and c is calculated by using Heron's formula, stated as

$$\text{Area of triangle} = \sqrt{s(s-a)(s-b)(s-c)}$$

where

$$s = \frac{a+b+c}{2}$$

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CHAPTER 11

SURFACE AREAS AND VOLUMES

11.1 Surface Area of a Right Circular Cone

We have already studied the surface areas of cube, cuboid and cylinder. We will now study the surface area of cone.

So far, we have been generating solids by stacking up congruent figures. Incidentally, such figures are called *prisms*. Now let us look at another kind of solid which is not a prism (These kinds of solids are called *pyramids*). Let us see how we can generate them.

Activity : Cut out a right-angled triangle ABC right angled at B. Paste a long thick string along one of the perpendicular sides say AB of the triangle [see Fig. 11.1(a)]. Hold the string with your hands on either sides of the triangle and rotate the triangle about the string a number of times. What happens? Do you recognize the shape that the triangle is forming as it rotates around the string [see Fig. 11.1(b)]? Does it remind you of the time you had eaten an ice-cream heaped into a container of that shape [see Fig. 11.1 (c) and (d)]?

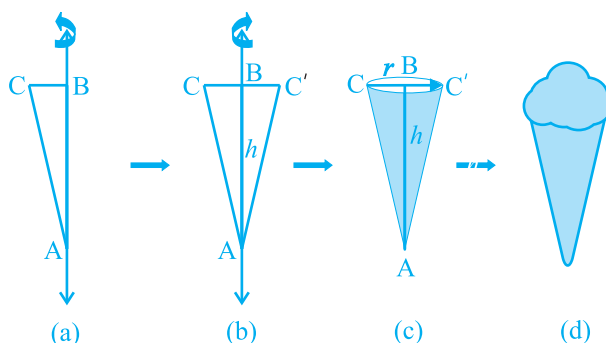


Fig. 11.1

This is called a *right circular cone*. In Fig. 11.1(c) of the right circular cone, the point A is called the vertex, AB is called the height, BC is called the *radius* and AC is called the slant height of the cone. Here B will be the centre of circular base of the cone. The height, radius and slant height of the cone are usually denoted by h , r and l respectively. Once again, let us see what kind of cone we can *not* call a right circular cone. Here, you are (see Fig. 11.2)! What you see in these figures are not right circular cones; because in (a), the line joining its vertex to the centre of its base is not at right angle to the base, and in (b) the base is not circular.

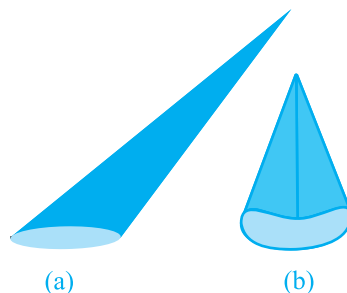


Fig. 11.2

As in the case of cylinder, since we will be studying only about right circular cones, remember that by ‘cone’ in this chapter, we shall mean a ‘right circular cone.’

Activity : (i) Cut out a neatly made paper cone that does not have any overlapped paper, straight along its side, and opening it out, to see the shape of paper that forms the surface of the cone. (The line along which you cut the cone is the *slant height* of the cone which is represented by l). It looks like a part of a round cake.

(ii) If you now bring the sides marked A and B at the tips together, you can see that the curved portion of Fig. 11.3 (c) will form the circular base of the cone.

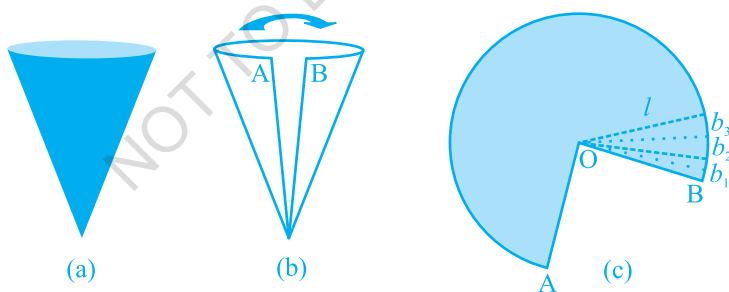


Fig. 11.3

(iii) If the paper like the one in Fig. 11.3 (c) is now cut into hundreds of little pieces, along the lines drawn from the point O, each cut portion is almost a small triangle, whose height is the slant height l of the cone.

(iv) Now the area of each triangle = $\frac{1}{2} \times \text{base of each triangle} \times l$.

So, area of the entire piece of paper

= sum of the areas of all the triangles

$$= \frac{1}{2}b_1l + \frac{1}{2}b_2l + \frac{1}{2}b_3l + \dots = \frac{1}{2}l(b_1 + b_2 + b_3 + \dots)$$

$$= \frac{1}{2} \times l \times \text{length of entire curved boundary of Fig. 11.3(c)}$$

(as $b_1 + b_2 + b_3 + \dots$ makes up the curved portion of the figure)

But the curved portion of the figure makes up the perimeter of the base of the cone and the circumference of the base of the cone $= 2\pi r$, where r is the base radius of the cone.

So, **Curved Surface Area of a Cone** $= \frac{1}{2} \times l \times 2\pi r = \pi rl$

where r is its base radius and l its slant height.

Note that $l^2 = r^2 + h^2$ (as can be seen from Fig. 11.4), by applying Pythagoras Theorem. Here h is the *height* of the cone.

Therefore, $l = \sqrt{r^2 + h^2}$

Now if the base of the cone is to be closed, then a circular piece of paper of radius r is also required whose area is πr^2 .

So, **Total Surface Area of a Cone** $= \pi rl + \pi r^2 = \pi r(l + r)$

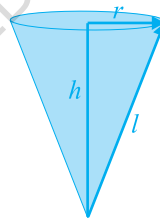


Fig. 11.4

Example 1 : Find the curved surface area of a right circular cone whose slant height is 10 cm and base radius is 7 cm.

Solution : Curved surface area $= \pi rl$

$$= \frac{22}{7} \times 7 \times 10 \text{ cm}^2$$

$$= 220 \text{ cm}^2$$

Example 2 : The height of a cone is 16 cm and its base radius is 12 cm. Find the curved surface area and the total surface area of the cone (Use $\pi = 3.14$).

Solution : Here, $h = 16$ cm and $r = 12$ cm.

So, from $l^2 = h^2 + r^2$, we have

$$l = \sqrt{16^2 + 12^2} \text{ cm} = 20 \text{ cm}$$

$$\begin{aligned}
 \text{So, curved surface area} &= \pi r l \\
 &= 3.14 \times 12 \times 20 \text{ cm}^2 \\
 &= 753.6 \text{ cm}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{Further, total surface area} &= \pi r l + \pi r^2 \\
 &= (753.6 + 3.14 \times 12 \times 12) \text{ cm}^2 \\
 &= (753.6 + 452.16) \text{ cm}^2 \\
 &= 1205.76 \text{ cm}^2
 \end{aligned}$$

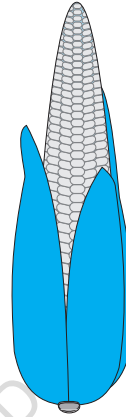


Fig. 11.5

Example 3 : A corn cob (see Fig. 11.5), shaped somewhat like a cone, has the radius of its broadest end as 2.1 cm and length (height) as 20 cm. If each 1 cm² of the surface of the cob carries an average of four grains, find how many grains you would find on the entire cob.

Solution : Since the grains of corn are found only on the curved surface of the corn cob, we would need to know the curved surface area of the corn cob to find the total number of grains on it. In this question, we are given the height of the cone, so we need to find its slant height.

$$\begin{aligned}
 \text{Here, } l &= \sqrt{r^2 + h^2} = \sqrt{(2.1)^2 + 20^2} \text{ cm} \\
 &= \sqrt{404.41} \text{ cm} = 20.11 \text{ cm}
 \end{aligned}$$

Therefore, the curved surface area of the corn cob = $\pi r l$

$$= \frac{22}{7} \times 2.1 \times 20.11 \text{ cm}^2 = 132.726 \text{ cm}^2 = 132.73 \text{ cm}^2 \text{ (approx.)}$$

Number of grains of corn on 1 cm² of the surface of the corn cob = 4

Therefore, number of grains on the entire curved surface of the cob

$$= 132.73 \times 4 = 530.92 = 531 \text{ (approx.)}$$

So, there would be approximately 531 grains of corn on the cob.

EXERCISE 11.1

Assume $\pi = \frac{22}{7}$, unless stated otherwise.

1. Diameter of the base of a cone is 10.5 cm and its slant height is 10 cm. Find its curved surface area.
2. Find the total surface area of a cone, if its slant height is 21 m and diameter of its base is 24 m.

3. Curved surface area of a cone is 308 cm^2 and its slant height is 14 cm. Find (i) radius of the base and (ii) total surface area of the cone.
4. A conical tent is 10 m high and the radius of its base is 24 m. Find (i) slant height of the tent.
(ii) cost of the canvas required to make the tent, if the cost of 1 m^2 canvas is ₹ 70.
5. What length of tarpaulin 3 m wide will be required to make conical tent of height 8 m and base radius 6 m? Assume that the extra length of material that will be required for stitching margins and wastage in cutting is approximately 20 cm (Use $\pi = 3.14$).
6. The slant height and base diameter of a conical tomb are 25 m and 14 m respectively. Find the cost of white-washing its curved surface at the rate of ₹ 210 per 100 m^2 .
7. A joker's cap is in the form of a right circular cone of base radius 7 cm and height 24 cm. Find the area of the sheet required to make 10 such caps.
8. A bus stop is barricaded from the remaining part of the road, by using 50 hollow cones made of recycled cardboard. Each cone has a base diameter of 40 cm and height 1 m. If the outer side of each of the cones is to be painted and the cost of painting is ₹ 12 per m^2 , what will be the cost of painting all these cones? (Use $\pi = 3.14$ and take $\sqrt{1.04} = 1.02$)

11.2 Surface Area of a Sphere

What is a sphere? Is it the same as a circle? Can you draw a circle on a paper? Yes, you can, because a circle is a plane closed figure whose every point lies at a constant distance (called **radius**) from a fixed point, which is called the **centre** of the circle. Now if you paste a string along a diameter of a circular disc and rotate it as you had rotated the triangle in the previous section, you see a new solid (see Fig 11.6). What does it resemble? A ball? Yes. It is called a **sphere**.

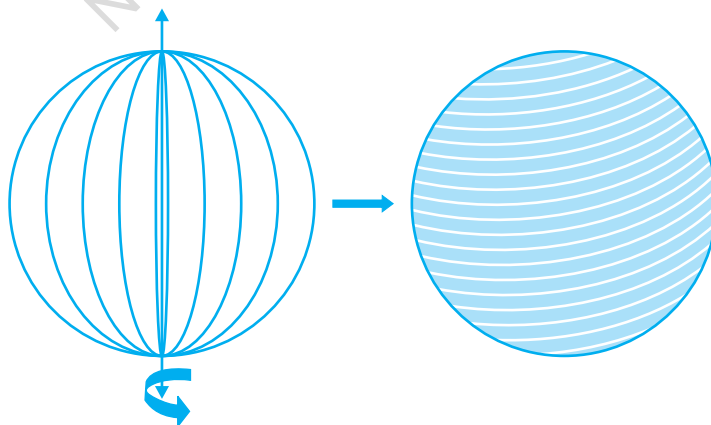


Fig. 11.6

Can you guess what happens to the centre of the circle, when it forms a sphere on rotation? Of course, it becomes the centre of the sphere. So, *a sphere is a three dimensional figure (solid figure), which is made up of all points in the space, which lie at a constant distance called the radius, from a fixed point called the centre of the sphere.*

Note : A sphere is like the surface of a ball. The word *solid sphere* is used for the solid whose surface is a sphere.

Activity : Have you ever played with a top or have you at least watched someone play with one? You must be aware of how a string is wound around it. Now, let us take a rubber ball and drive a nail into it. Taking support of the nail, let us wind a string around the ball. When you have reached the ‘fullest’ part of the ball, use pins to keep the string in place, and continue to wind the string around the remaining part of the ball, till you have completely covered the ball [see Fig. 11.7(a)]. Mark the starting and finishing points on the string, and slowly unwind the string from the surface of the ball.

Now, ask your teacher to help you in measuring the diameter of the ball, from which you easily get its radius. Then on a sheet of paper, draw four circles with radius equal to the radius of the ball. Start filling the circles one by one, with the string you had wound around the ball [see Fig. 11.7(b)].

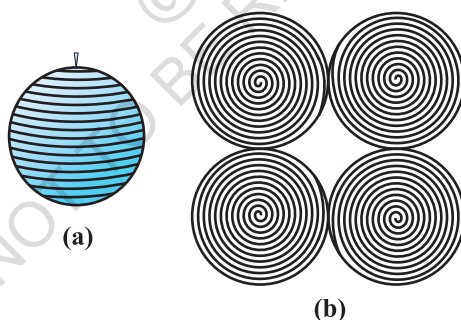


Fig. 11.7

What have you achieved in all this?

The string, which had completely covered the surface area of the sphere, has been used to completely fill the regions of four circles, all of the same radius as of the sphere.

So, what does that mean? This suggests that the surface area of a sphere of radius r

$$= 4 \text{ times the area of a circle of radius } r = 4 \times (\pi r^2)$$

So,

Surface Area of a Sphere = $4 \pi r^2$
--

where r is the radius of the sphere.

How many faces do you see in the surface of a sphere? There is only one, which is curved.

Now, let us take a solid sphere, and slice it exactly ‘through the middle’ with a plane that passes through its centre. What happens to the sphere?

Yes, it gets divided into two equal parts (see Fig. 11.8)! What will each half be called? It is called a **hemisphere**. (Because ‘hemi’ also means ‘half’)



Fig. 11.8

And what about the surface of a hemisphere? How many faces does it have?

Two! There is a curved face and a flat face (base).

The curved surface area of a hemisphere is half the surface area of the sphere, which is $\frac{1}{2}$ of $4\pi r^2$.

Therefore, **Curved Surface Area of a Hemisphere = $2\pi r^2$**

where r is the radius of the sphere of which the hemisphere is a part.

Now taking the two faces of a hemisphere, its surface area $2\pi r^2 + \pi r^2$

So, **Total Surface Area of a Hemisphere = $3\pi r^2$**

Example 4 : Find the surface area of a sphere of radius 7 cm.

Solution : The surface area of a sphere of radius 7 cm would be

$$4\pi r^2 = 4 \times \frac{22}{7} \times 7 \times 7 \text{ cm}^2 = 616 \text{ cm}^2$$

Example 5 : Find (i) the curved surface area and (ii) the total surface area of a hemisphere of radius 21 cm.

Solution : The curved surface area of a hemisphere of radius 21 cm would be

$$= 2\pi r^2 = 2 \times \frac{22}{7} \times 21 \times 21 \text{ cm}^2 = 2772 \text{ cm}^2$$

(ii) the total surface area of the hemisphere would be

$$3\pi r^2 = 3 \times \frac{22}{7} \times 21 \times 21 \text{ cm}^2 = 4158 \text{ cm}^2$$

Example 6 : The hollow sphere, in which the circus motorcyclist performs his stunts, has a diameter of 7 m. Find the area available to the motorcyclist for riding.

Solution : Diameter of the sphere = 7 m. Therefore, radius is 3.5 m. So, the riding space available for the motorcyclist is the surface area of the 'sphere' which is given by

$$\begin{aligned} 4\pi r^2 &= 4 \times \frac{22}{7} \times 3.5 \times 3.5 \text{ m}^2 \\ &= 154 \text{ m}^2 \end{aligned}$$

Example 7 : A hemispherical dome of a building needs to be painted (see Fig. 11.9). If the circumference of the base of the dome is 17.6 m, find the cost of painting it, given the cost of painting is ₹ 5 per 100 cm².

Solution : Since only the rounded surface of the dome is to be painted, we would need to find the curved surface area of the hemisphere to know the extent of painting that needs to be done. Now, circumference of the dome = 17.6 m. Therefore, $17.6 = 2\pi r$.

$$\text{So, the radius of the dome} = 17.6 \times \frac{7}{2 \times 22} \text{ m} = 2.8 \text{ m}$$

The curved surface area of the dome = $2\pi r^2$

$$\begin{aligned} &= 2 \times \frac{22}{7} \times 2.8 \times 2.8 \text{ m}^2 \\ &= 49.28 \text{ m}^2 \end{aligned}$$

Now, cost of painting 100 cm² is ₹ 5.

So, cost of painting 1 m² = ₹ 500

Therefore, cost of painting the whole dome

$$\begin{aligned} &= ₹ 500 \times 49.28 \\ &= ₹ 24640 \end{aligned}$$

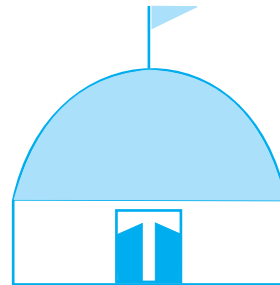


Fig. 11.9

EXERCISE 11.2

Assume $\pi = \frac{22}{7}$, unless stated otherwise.

1. Find the surface area of a sphere of radius:

(i) 10.5 cm

(ii) 5.6 cm

(iii) 14 cm

2. Find the surface area of a sphere of diameter:
 - (i) 14 cm
 - (ii) 21 cm
 - (iii) 3.5 m
3. Find the total surface area of a hemisphere of radius 10 cm. (Use $\pi = 3.14$)
4. The radius of a spherical balloon increases from 7 cm to 14 cm as air is being pumped into it. Find the ratio of surface areas of the balloon in the two cases.
5. A hemispherical bowl made of brass has inner diameter 10.5 cm. Find the cost of tin-plating it on the inside at the rate of ₹ 16 per 100 cm².
6. Find the radius of a sphere whose surface area is 154 cm².
7. The diameter of the moon is approximately one fourth of the diameter of the earth. Find the ratio of their surface areas.
8. A hemispherical bowl is made of steel, 0.25 cm thick. The inner radius of the bowl is 5 cm. Find the outer curved surface area of the bowl.
9. A right circular cylinder just encloses a sphere of radius r (see Fig. 11.10). Find
 - (i) surface area of the sphere,
 - (ii) curved surface area of the cylinder,
 - (iii) ratio of the areas obtained in (i) and (ii).

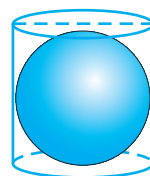


Fig. 11.10

11.3 Volume of a Right Circular Cone

In earlier classes we have studied the volumes of cube, cuboid and cylinder

In Fig 11.11, can you see that there is a right circular cylinder and a right circular cone of the same base radius and the same height?

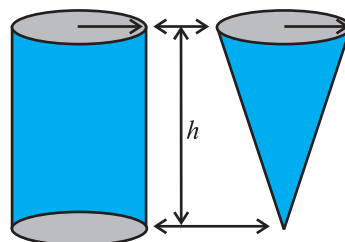


Fig. 11.11

Activity : Try to make a hollow cylinder and a hollow cone like this with the same base radius and the same height (see Fig. 11.11). Then, we can try out an experiment that will help us, to see practically what the volume of a right circular cone would be!

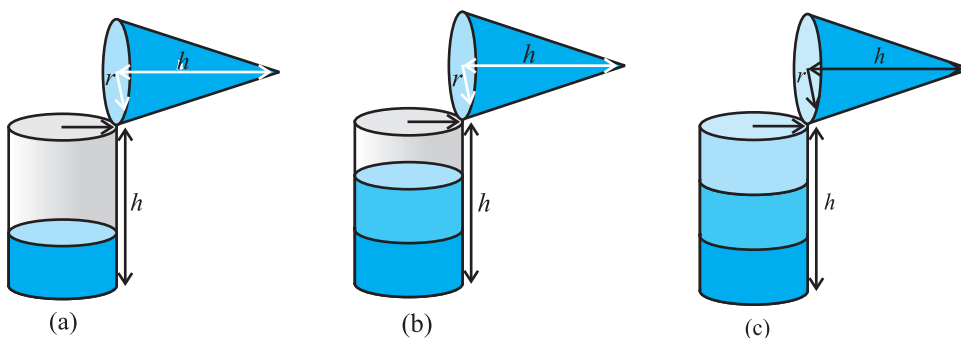


Fig. 11.12

So, let us start like this.

Fill the cone up to the brim with sand once, and empty it into the cylinder. We find that it fills up only a part of the cylinder [see Fig. 11.12(a)].

When we fill up the cone again to the brim, and empty it into the cylinder, we see that the cylinder is still not full [see Fig. 11.12(b)].

When the cone is filled up for the third time, and emptied into the cylinder, it can be seen that the cylinder is also full to the brim [see Fig. 11.12(c)].

With this, we can safely come to the conclusion that three times the volume of a cone, makes up the volume of a cylinder, which has the same base radius and the same height as the cone, which means that the volume of the cone is one-third the volume of the cylinder.

So,
$$\text{Volume of a Cone} = \frac{1}{3} \pi r^2 h$$

where r is the base radius and h is the height of the cone.

Example 8 : The height and the slant height of a cone are 21 cm and 28 cm respectively. Find the volume of the cone.

Solution : From $l^2 = r^2 + h^2$, we have

$$r = \sqrt{l^2 - h^2} = \sqrt{28^2 - 21^2} \text{ cm} = 7\sqrt{7} \text{ cm}$$

$$\begin{aligned} \text{So, volume of the cone} &= \frac{1}{3} \pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times 7\sqrt{7} \times 7\sqrt{7} \times 21 \text{ cm}^3 \\ &= 7546 \text{ cm}^3 \end{aligned}$$

Example 9 : Monica has a piece of canvas whose area is 551 m^2 . She uses it to have a conical tent made, with a base radius of 7 m. Assuming that all the stitching margins and the wastage incurred while cutting, amounts to approximately 1 m^2 , find the volume of the tent that can be made with it.

Solution : Since the area of the canvas = 551 m^2 and area of the canvas lost in wastage is 1 m^2 , therefore the area of canvas available for making the tent is $(551 - 1) \text{ m}^2 = 550 \text{ m}^2$.

Now, the surface area of the tent = 550 m^2 and the required base radius of the conical tent = 7 m

Note that a tent has only a curved surface (the floor of a tent is not covered by canvas!!).

Therefore, curved surface area of tent = 550 m^2 .

That is, $\pi r l = 550$

or,
$$\frac{22}{7} \times 7 \times l = 550$$

or,
$$l = 3 \frac{550}{22} \text{ m} = 25 \text{ m}$$

Now,
$$l^2 = r^2 + h^2$$

Therefore,
$$h = \sqrt{l^2 - r^2} = \sqrt{25^2 - 7^2} \text{ m} = \sqrt{625 - 49} \text{ m} = \sqrt{576} \text{ m}$$
$$= 24 \text{ m}$$

So, the volume of the conical tent = $\frac{1}{3} \pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times 24 \text{ m}^3 = 1232 \text{ m}^3$.

EXERCISE 11.3

Assume $\pi = \frac{22}{7}$, unless stated otherwise.

- Find the volume of the right circular cone with
 - radius 6 cm, height 7 cm
 - radius 3.5 cm, height 12 cm
- Find the capacity in litres of a conical vessel with
 - radius 7 cm, slant height 25 cm
 - height 12 cm, slant height 13 cm
- The height of a cone is 15 cm. If its volume is 1570 cm^3 , find the radius of the base. (Use $\pi = 3.14$)
- If the volume of a right circular cone of height 9 cm is $48 \pi \text{ cm}^3$, find the diameter of its base.
- A conical pit of top diameter 3.5 m is 12 m deep. What is its capacity in kilolitres?
- The volume of a right circular cone is 9856 cm^3 . If the diameter of the base is 28 cm, find
 - height of the cone
 - slant height of the cone
 - curved surface area of the cone
- A right triangle ABC with sides 5 cm, 12 cm and 13 cm is revolved about the side 12 cm. Find the volume of the solid so obtained.
- If the triangle ABC in the Question 7 above is revolved about the side 5 cm, then find the volume of the solid so obtained. Find also the ratio of the volumes of the two solids obtained in Questions 7 and 8.
- A heap of wheat is in the form of a cone whose diameter is 10.5 m and height is 3 m. Find its volume. The heap is to be covered by canvas to protect it from rain. Find the area of the canvas required.

11.4 Volume of a Sphere

Now, let us see how to go about measuring the volume of a sphere. First, take two or three spheres of different radii, and a container big enough to be able to put each of the spheres into it, one at a time. Also, take a large trough in which you can place the container. Then, fill the container up to the brim with water [see Fig. 11.13(a)].

Now, carefully place one of the spheres in the container. Some of the water from the container will over flow into the trough in which it is kept [see Fig. 11.13(b)]. Carefully pour out the water from the trough into a measuring cylinder (i.e., a graduated cylindrical jar) and measure the water over flowed [see Fig. 11.13(c)]. Suppose the radius of the immersed sphere is r (you can find the radius by measuring the diameter of the sphere). Then evaluate $\frac{4}{3} \pi r^3$. Do you find this value almost equal to the measure of the volume over flowed?

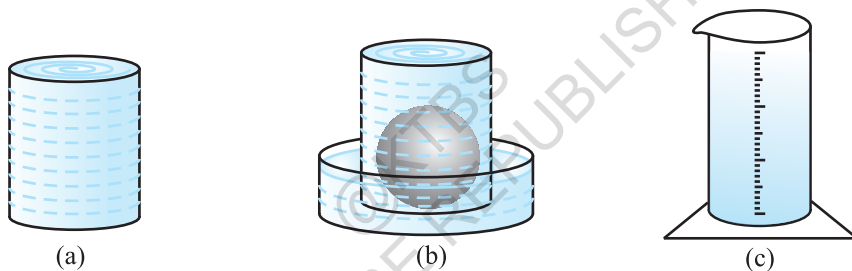


Fig. 11.13

Once again repeat the procedure done just now, with a different size of sphere. Find the radius R of this sphere and then calculate the value of $\frac{4}{3} \pi R^3$. Once again this value is nearly equal to the measure of the volume of the water displaced (over flowed) by the sphere. What does this tell us? We know that the volume of the sphere is the same as the measure of the volume of the water displaced by it. By doing this experiment repeatedly with spheres of varying radii, we are getting the same result, namely, the volume of a sphere is equal to $\frac{4}{3} \pi$ times the cube of its radius. This gives us the idea that

$$\text{Volume of a Sphere} = \frac{4}{3} \pi r^3$$

where r is the radius of the sphere.

Later, in higher classes it can be proved also. But at this stage, we will just take it as true.

Since a hemisphere is half of a sphere, can you guess what the volume of a hemisphere will be? Yes, it is $\frac{1}{2}$ of $\frac{4}{3} \pi r^3 = \frac{2}{3} \pi r^3$.

So, **Volume of a Hemisphere** = $\frac{2}{3} \pi r^3$

where r is the radius of the hemisphere.

Let us take some examples to illustrate the use of these formulae.

Example 10 : Find the volume of a sphere of radius 11.2 cm.

Solution : Required volume = $\frac{4}{3} \pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 11.2 \times 11.2 \times 11.2 \text{ cm}^3 = 5887.32 \text{ cm}^3$$

Example 11 : A shot-putt is a metallic sphere of radius 4.9 cm. If the density of the metal is 7.8 g per cm^3 , find the mass of the shot-putt.

Solution : Since the shot-putt is a solid sphere made of metal and its mass is equal to the product of its volume and density, we need to find the volume of the sphere.

Now, volume of the sphere = $\frac{4}{3} \pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 4.9 \times 4.9 \times 4.9 \text{ cm}^3$$

$$= 493 \text{ cm}^3 \text{ (nearly)}$$

Further, mass of 1 cm^3 of metal is 7.8 g.

Therefore, mass of the shot-putt = $7.8 \times 493 \text{ g}$

$$= 3845.44 \text{ g} = 3.85 \text{ kg (nearly)}$$

Example 12 : A hemispherical bowl has a radius of 3.5 cm. What would be the volume of water it would contain?

Solution : The volume of water the bowl can contain

$$= \frac{2}{3} \pi r^3$$

$$= \frac{2}{3} \times \frac{22}{7} \times 3.5 \times 3.5 \times 3.5 \text{ cm}^3 = 89.8 \text{ cm}^3$$

EXERCISE 11.4

Assume $\pi = \frac{22}{7}$, unless stated otherwise.

- Find the volume of a sphere whose radius is
(i) 7 cm (ii) 0.63 m
- Find the amount of water displaced by a solid spherical ball of diameter
(i) 28 cm (ii) 0.21 m
- The diameter of a metallic ball is 4.2 cm. What is the mass of the ball, if the density of the metal is 8.9 g per cm^3 ?
- The diameter of the moon is approximately one-fourth of the diameter of the earth. What fraction of the volume of the earth is the volume of the moon?
- How many litres of milk can a hemispherical bowl of diameter 10.5 cm hold?
- A hemispherical tank is made up of an iron sheet 1 cm thick. If the inner radius is 1 m, then find the volume of the iron used to make the tank.
- Find the volume of a sphere whose surface area is 154 cm^2 .
- A dome of a building is in the form of a hemisphere. From inside, it was white-washed at the cost of ₹ 4989.60. If the cost of white-washing is ₹ 20 per square metre, find the
(i) inside surface area of the dome, (ii) volume of the air inside the dome.
- Twenty seven solid iron spheres, each of radius r and surface area S are melted to form a sphere with surface area S' . Find the
(i) radius r' of the new sphere, (ii) ratio of S and S' .
- A capsule of medicine is in the shape of a sphere of diameter 3.5 mm. How much medicine (in mm^3) is needed to fill this capsule?

11.5 Summary

In this chapter, you have studied the following points:

- Curved surface area of a cone $= \pi r l$
- Total surface area of a right circular cone $= \pi r l + \pi r^2$, i.e., $\pi r (l + r)$
- Surface area of a sphere of radius $r = 4 \pi r^2$
- Curved surface area of a hemisphere $= 2 \pi r^2$
- Total surface area of a hemisphere $= 3 \pi r^2$
- Volume of a cone $= \frac{1}{3} \pi r^2 h$
- Volume of a sphere of radius $r = \frac{4}{3} \pi r^3$
- Volume of a hemisphere $= \frac{2}{3} \pi r^3$

[Here, letters l , b , h , a , r , etc. have been used in their usual meaning, depending on the context.]



0962CH14

CHAPTER 12

STATISTICS

12.1 Graphical Representation of Data

The representation of data by tables has already been discussed. Now let us turn our attention to another representation of data, i.e., the graphical representation. It is well said that one picture is better than a thousand words. Usually comparisons among the individual items are best shown by means of graphs. The representation then becomes easier to understand than the actual data. We shall study the following graphical representations in this section.

- (A) Bar graphs
- (B) Histograms of uniform width, and of varying widths
- (C) Frequency polygons

(A) Bar Graphs

In earlier classes, you have already studied and constructed bar graphs. Here we shall discuss them through a more formal approach. Recall that a bar graph is a pictorial representation of data in which usually bars of uniform width are drawn with equal spacing between them on one axis (say, the x -axis), depicting the variable. The values of the variable are shown on the other axis (say, the y -axis) and the heights of the bars depend on the values of the variable.

Example 1 : In a particular section of Class IX, 40 students were asked about the months of their birth and the following graph was prepared for the data so obtained:

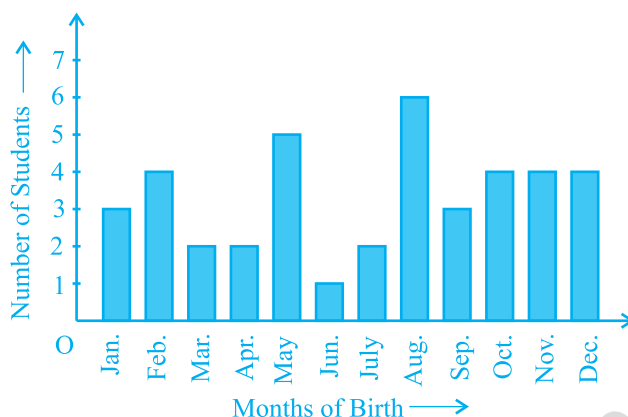


Fig. 12.1

Observe the bar graph given above and answer the following questions:

- How many students were born in the month of November?
- In which month were the maximum number of students born?

Solution : Note that the variable here is the ‘month of birth’, and the value of the variable is the ‘Number of students born’.

- 4 students were born in the month of November.
- The Maximum number of students were born in the month of August.

Let us now recall how a bar graph is constructed by considering the following example.

Example 2 : A family with a monthly income of ₹ 20,000 had planned the following expenditures per month under various heads:

Table 12.1

Heads	Expenditure (in thousand rupees)
Grocery	4
Rent	5
Education of children	5
Medicine	2
Fuel	2
Entertainment	1
Miscellaneous	1

Draw a bar graph for the data above.

Solution : We draw the bar graph of this data in the following steps. Note that the unit in the second column is thousand rupees. So, '4' against 'grocery' means ₹4000.

1. We represent the Heads (variable) on the horizontal axis choosing any scale, since the width of the bar is not important. But for clarity, we take equal widths for all bars and maintain equal gaps in between. Let one Head be represented by one unit.
2. We represent the expenditure (value) on the vertical axis. Since the maximum expenditure is ₹5000, we can choose the scale as 1 unit = ₹1000.
3. To represent our first Head, i.e., grocery, we draw a rectangular bar with width 1 unit and height 4 units.
4. Similarly, other Heads are represented leaving a gap of 1 unit in between two consecutive bars.

The bar graph is drawn in Fig. 12.2.

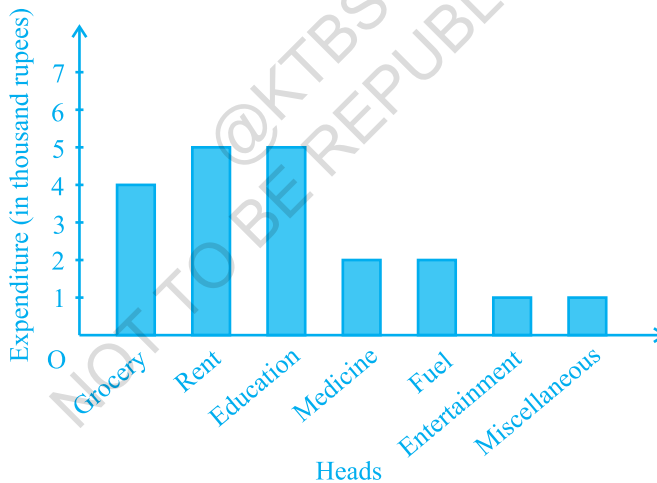


Fig. 12.2

Here, you can easily visualise the relative characteristics of the data at a glance, e.g., the expenditure on education is more than double that of medical expenses. Therefore, in some ways it serves as a better representation of data than the tabular form.

Activity 1 : Continuing with the same four groups of Activity 1, represent the data by suitable bar graphs.

Let us now see how a frequency distribution table for *continuous* class intervals can be represented graphically.

(B) Histogram

This is a form of representation like the bar graph, but it is used for continuous class intervals. For instance, consider the frequency distribution Table 12.2, representing the weights of 36 students of a class:

Table 12.2

Weights (in kg)	Number of students
30.5 - 35.5	9
35.5 - 40.5	6
40.5 - 45.5	15
45.5 - 50.5	3
50.5 - 55.5	1
55.5 - 60.5	2
Total	36

Let us represent the data given above graphically as follows:

- We represent the weights on the horizontal axis on a suitable scale. We can choose the scale as 1 cm = 5 kg. Also, since the first class interval is starting from 30.5 and not zero, we show it on the graph by marking a *kink* or a break on the axis.
- We represent the number of students (frequency) on the vertical axis on a suitable scale. Since the maximum frequency is 15, we need to choose the scale to accommodate this maximum frequency.
- We now draw rectangles (or rectangular bars) of width equal to the class-size and lengths according to the frequencies of the corresponding class intervals. For example, the rectangle for the class interval 30.5 - 35.5 will be of width 1 cm and length 4.5 cm.
- In this way, we obtain the graph as shown in Fig. 12.3:

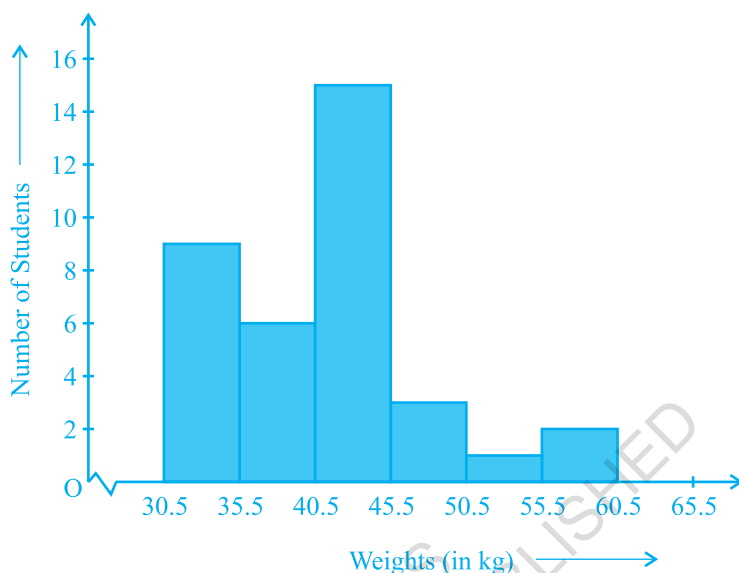


Fig. 12.3

Observe that since there are no gaps in between consecutive rectangles, the resultant graph appears like a solid figure. This is called a *histogram*, which is a graphical representation of a grouped frequency distribution with continuous classes. Also, unlike a bar graph, the width of the bar plays a significant role in its construction.

Here, in fact, areas of the rectangles erected are proportional to the corresponding frequencies. However, since the widths of the rectangles are all equal, the lengths of the rectangles are proportional to the frequencies. That is why, we draw the lengths according to (iii) above.

Now, consider a situation different from the one above.

Example 3 : A teacher wanted to analyse the performance of two sections of students in a mathematics test of 100 marks. Looking at their performances, she found that a few students got under 20 marks and a few got 70 marks or above. So she decided to group them into intervals of varying sizes as follows: 0 - 20, 20 - 30, . . . , 60 - 70, 70 - 100. Then she formed the following table:

Table 12.3

Marks	Number of students
0 - 20	7
20 - 30	10
30 - 40	10
40 - 50	20
50 - 60	20
60 - 70	15
70 - above	8
Total	90

A histogram for this table was prepared by a student as shown in Fig. 12.4.

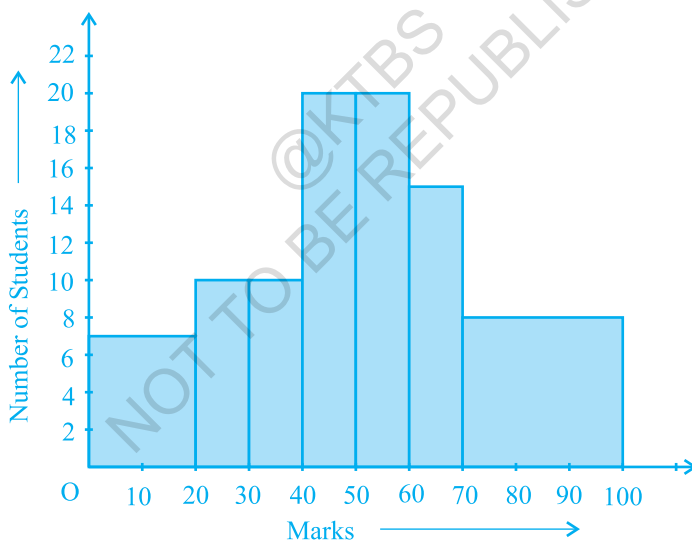


Fig. 12.4

Carefully examine this graphical representation. Do you think that it correctly represents the data? No, the graph is giving us a misleading picture. As we have mentioned earlier, the areas of the rectangles are proportional to the frequencies in a histogram. Earlier this problem did not arise, because the widths of all the rectangles were equal. But here, since the widths of the rectangles are varying, the histogram above does not

give a correct picture. For example, it shows a greater frequency in the interval 70 - 100, than in 60 - 70, which is not the case.

So, we need to make certain modifications in the lengths of the rectangles so that the areas are again proportional to the frequencies.

The steps to be followed are as given below:

1. Select a class interval with the minimum class size. In the example above, the minimum class-size is 10.
2. The lengths of the rectangles are then modified to be proportionate to the class-size 10.

For instance, when the class-size is 20, the length of the rectangle is 7. So when the class-size is 10, the length of the rectangle will be $\frac{7}{20} \times 10 = 3.5$.

Similarly, proceeding in this manner, we get the following table:

Table 12.4

Marks	Frequency	Width of the class	Length of the rectangle
0 - 20	7	20	$\frac{7}{20} \times 10 = 3.5$
20 - 30	10	10	$\frac{10}{10} \times 10 = 10$
30 - 40	10	10	$\frac{10}{10} \times 10 = 10$
40 - 50	20	10	$\frac{20}{10} \times 10 = 20$
50 - 60	20	10	$\frac{20}{10} \times 10 = 20$
60 - 70	15	10	$\frac{15}{10} \times 10 = 15$
70 - 100	8	30	$\frac{8}{30} \times 10 = 2.67$

Since we have calculated these lengths for an interval of 10 marks in each case, we may call these lengths as “proportion of students per 10 marks interval”.

So, the correct histogram with varying width is given in Fig. 12.5.

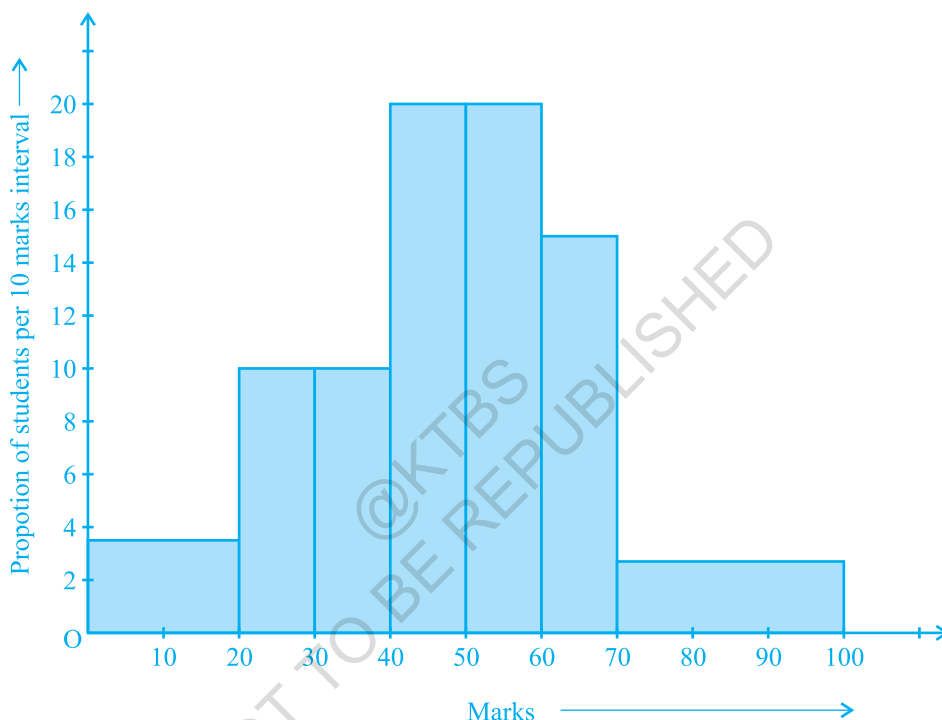


Fig. 12.5

(C) Frequency Polygon

There is yet another visual way of representing quantitative data and its frequencies. This is a polygon. To see what we mean, consider the histogram represented by Fig. 12.3. Let us join the mid-points of the upper sides of the adjacent rectangles of this histogram by means of line segments. Let us call these mid-points B, C, D, E, F and G. When joined by line segments, we obtain the figure BCDEFG (see Fig. 12.6). To complete the polygon, we assume that there is a class interval with frequency zero

before 30.5 - 35.5, and one after 55.5 - 60.5, and their mid-points are A and H, respectively. ABCDEFGH is the frequency polygon corresponding to the data shown in Fig. 12.3. We have shown this in Fig. 12.6.

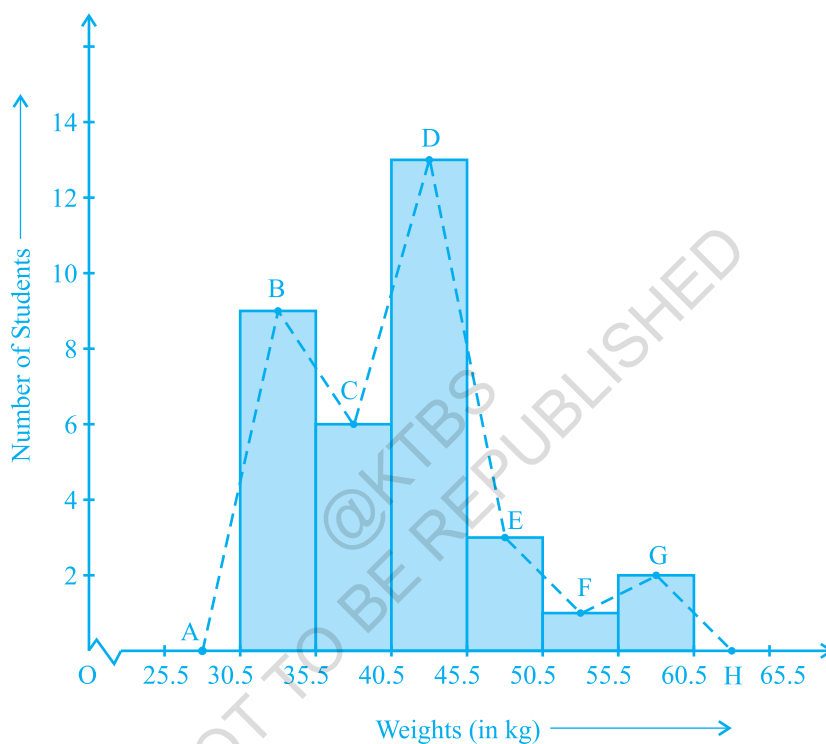


Fig. 12.6

Although, there exists no class preceding the lowest class and no class succeeding the highest class, addition of the two class intervals with zero frequency enables us to make the area of the frequency polygon the same as the area of the histogram. Why is this so? (**Hint** : Use the properties of congruent triangles.)

Now, the question arises: how do we complete the polygon when there is no class preceding the first class? Let us consider such a situation.

Example 4 : Consider the marks, out of 100, obtained by 51 students of a class in a test, given in Table 12.5.

Table 12.5

Marks	Number of students
0 - 10	5
10 - 20	10
20 - 30	4
30 - 40	6
40 - 50	7
50 - 60	3
60 - 70	2
70 - 80	2
80 - 90	3
90 - 100	9
Total	51

Draw a frequency polygon corresponding to this frequency distribution table.

Solution : Let us first draw a histogram for this data and mark the mid-points of the tops of the rectangles as B, C, D, E, F, G, H, I, J, K, respectively. Here, the first class is 0-10. So, to find the class preceeding 0-10, we extend the horizontal axis in the negative direction and find the mid-point of the imaginary class-interval $(-10) - 0$. The first end point, i.e., B is joined to this mid-point with zero frequency on the negative direction of the horizontal axis. The point where this line segment meets the vertical axis is marked as A. Let L be the mid-point of the class succeeding the last class of the given data. Then OABCDEFGHJKLM is the frequency polygon, which is shown in Fig. 12.7.

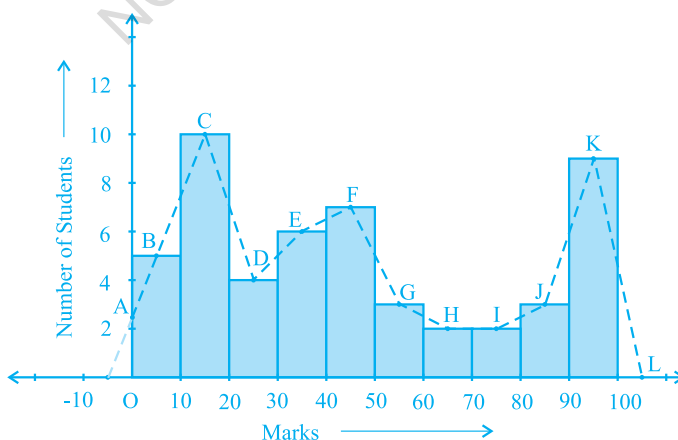


Fig. 12.7

Frequency polygons can also be drawn independently without drawing histograms. For this, we require the mid-points of the class-intervals used in the data. These mid-points of the class-intervals are called **class-marks**.

To find the class-mark of a class interval, we find the sum of the upper limit and lower limit of a class and divide it by 2. Thus,

$$\text{Class-mark} = \frac{\text{Upper limit} + \text{Lower limit}}{2}$$

Let us consider an example.

Example 5 : In a city, the weekly observations made in a study on the cost of living index are given in the following table:

Table 12.6

Cost of living index	Number of weeks
140 - 150	5
150 - 160	10
160 - 170	20
170 - 180	9
180 - 190	6
190 - 200	2
Total	52

Draw a frequency polygon for the data above (without constructing a histogram).

Solution : Since we want to draw a frequency polygon without a histogram, let us find the class-marks of the classes given above, that is of 140 - 150, 150 - 160,....

For 140 - 150, the upper limit = 150, and the lower limit = 140

$$\text{So, the class-mark} = \frac{150 + 140}{2} = \frac{290}{2} = 145.$$

Continuing in the same manner, we find the class-marks of the other classes as well. So, the new table obtained is as shown in the following table:

Table 12.7

Classes	Class-marks	Frequency
140 - 150	145	5
150 - 160	155	10
160 - 170	165	20
170 - 180	175	9
180 - 190	185	6
190 - 200	195	2
Total		52

We can now draw a frequency polygon by plotting the class-marks along the horizontal axis, the frequencies along the vertical-axis, and then plotting and joining the points B(145, 5), C(155, 10), D(165, 20), E(175, 9), F(185, 6) and G(195, 2) by line segments. We should not forget to plot the point corresponding to the class-mark of the class 130 - 140 (just before the lowest class 140 - 150) with zero frequency, that is, A(135, 0), and the point H (205, 0) occurs immediately after G(195, 2). So, the resultant frequency polygon will be ABCDEFGH (see Fig. 12.8).

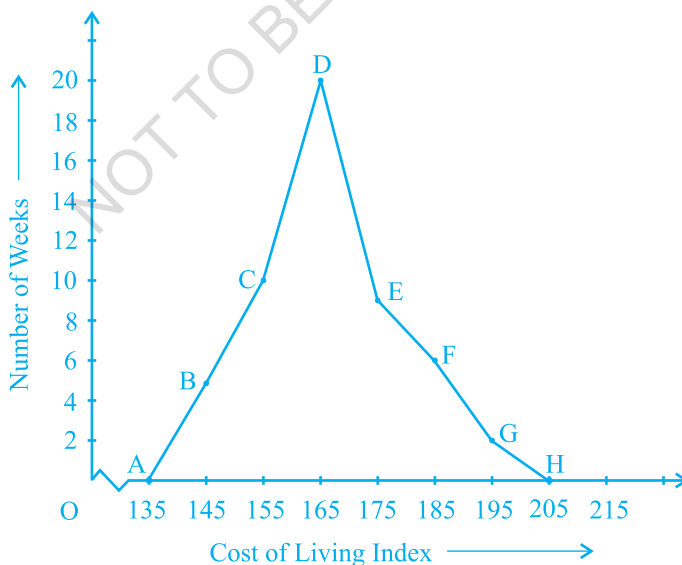


Fig. 12.8

Frequency polygons are used when the data is continuous and very large. It is very useful for comparing two different sets of data of the same nature, for example, comparing the performance of two different sections of the same class.

EXERCISE 12.1

1. A survey conducted by an organisation for the cause of illness and death among the women between the ages 15 - 44 (in years) worldwide, found the following figures (in %):

S.No.	Causes	Female fatality rate (%)
1.	Reproductive health conditions	31.8
2.	Neuropsychiatric conditions	25.4
3.	Injuries	12.4
4.	Cardiovascular conditions	4.3
5.	Respiratory conditions	4.1
6.	Other causes	22.0

- (i) Represent the information given above graphically.
 - (ii) Which condition is the major cause of women's ill health and death worldwide?
 - (iii) Try to find out, with the help of your teacher, any two factors which play a major role in the cause in (ii) above being the major cause.
2. The following data on the number of girls (to the nearest ten) per thousand boys in different sections of Indian society is given below.

Section	Number of girls per thousand boys
Scheduled Caste (SC)	940
Scheduled Tribe (ST)	970
Non SC/ST	920
Backward districts	950
Non-backward districts	920
Rural	930
Urban	910

- (i) Represent the information above by a bar graph.
- (ii) In the classroom discuss what conclusions can be arrived at from the graph.
3. Given below are the seats won by different political parties in the polling outcome of a state assembly elections:

Political Party	A	B	C	D	E	F
Seats Won	75	55	37	29	10	37

- (i) Draw a bar graph to represent the polling results.
- (ii) Which political party won the maximum number of seats?
4. The length of 40 leaves of a plant are measured correct to one millimetre, and the obtained data is represented in the following table:

Length (in mm)	Number of leaves
118 - 126	3
127 - 135	5
136 - 144	9
145 - 153	12
154 - 162	5
163 - 171	4
172 - 180	2

- (i) Draw a histogram to represent the given data. [Hint: First make the class intervals continuous]
- (ii) Is there any other suitable graphical representation for the same data?
- (iii) Is it correct to conclude that the maximum number of leaves are 153 mm long? Why?
5. The following table gives the life times of 400 neon lamps:

Life time (in hours)	Number of lamps
300 - 400	14
400 - 500	56
500 - 600	60
600 - 700	86
700 - 800	74
800 - 900	62
900 - 1000	48

- (i) Represent the given information with the help of a histogram.
- (ii) How many lamps have a life time of more than 700 hours?
6. The following table gives the distribution of students of two sections according to the marks obtained by them:

Section A		Section B	
Marks	Frequency	Marks	Frequency
0 - 10	3	0 - 10	5
10 - 20	9	10 - 20	19
20 - 30	17	20 - 30	15
30 - 40	12	30 - 40	10
40 - 50	9	40 - 50	1

Represent the marks of the students of both the sections on the same graph by two frequency polygons. From the two polygons compare the performance of the two sections.

7. The runs scored by two teams A and B on the first 60 balls in a cricket match are given below:

Number of balls	Team A	Team B
1 - 6	2	5
7 - 12	1	6
13 - 18	8	2
19 - 24	9	10
25 - 30	4	5
31 - 36	5	6
37 - 42	6	3
43 - 48	10	4
49 - 54	6	8
55 - 60	2	10

Represent the data of both the teams on the same graph by frequency polygons.

[Hint : First make the class intervals continuous.]

8. A random survey of the number of children of various age groups playing in a park was found as follows:

Age (in years)	Number of children
1 - 2	5
2 - 3	3
3 - 5	6
5 - 7	12
7 - 10	9
10 - 15	10
15 - 17	4

Draw a histogram to represent the data above.

9. 100 surnames were randomly picked up from a local telephone directory and a frequency distribution of the number of letters in the English alphabet in the surnames was found as follows:

Number of letters	Number of surnames
1 - 4	6
4 - 6	30
6 - 8	44
8 - 12	16
12 - 20	4

- Draw a histogram to depict the given information.
- Write the class interval in which the maximum number of surnames lie.

12.2 Summary

In this chapter, you have studied the following points:

- How data can be presented graphically in the form of bar graphs, histograms and frequency polygons.

INTRODUCTION TO MATHEMATICAL MODELLING

A2.1 Introduction

Right from your earlier classes, you have been solving problems related to the real-world around you. For example, you have solved problems in simple interest using the formula for finding it. The formula (or equation) is a relation between the interest and the other three quantities that are related to it, the principal, the rate of interest and the period. This formula is an example of a **mathematical model**. A **mathematical model** is a mathematical relation that describes some real-life situation.

Mathematical models are used to solve many real-life situations like:

- launching a satellite.
- predicting the arrival of the monsoon.
- controlling pollution due to vehicles.
- reducing traffic jams in big cities.

In this chapter, we will introduce you to the process of constructing mathematical models, which is called **mathematical modelling**. In mathematical modelling, we take a real-world problem and write it as an equivalent mathematical problem. We then solve the mathematical problem, and interpret its solution in terms of the real-world problem. After this we see to what extent the solution is valid in the context of the real-world problem. So, the *stages* involved in mathematical modelling are formulation, solution, interpretation and validation.

We will start by looking at the process you undertake when solving word problems, in Section A2.2. Here, we will discuss some word problems that are similar to the ones you have solved in your earlier classes. We will see later that the steps that are used for solving word problems are some of those used in mathematical modelling also.

In the next section, that is Section A2.3, we will discuss some simple models.

In Section A2.4, we will discuss the overall process of modelling, its advantages and some of its limitations.

A2.2 Review of Word Problems

In this section, we will discuss some word problems that are similar to the ones that you have solved in your earlier classes. Let us start with a problem on direct variation.

Example 1 : I travelled 432 kilometres on 48 litres of petrol in my car. I have to go by my car to a place which is 180 km away. How much petrol do I need?

Solution : We will list the steps involved in solving the problem.

Step 1 : Formulation : You know that farther we travel, the more petrol we require, that is, the amount of petrol we need varies directly with the distance we travel.

Petrol needed for travelling 432 km = 48 litres

Petrol needed for travelling 180 km = ?

Mathematical Description : Let

x = distance I travel

y = petrol I need

y varies directly with x .

So, $y = kx$, where k is a constant.

I can travel 432 kilometres with 48 litres of petrol.

So, $y = 48, x = 432$.

Therefore, $k = \frac{y}{x} = \frac{48}{432} = \frac{1}{9}$.

Since $y = kx$,

therefore, $y = \frac{1}{9}x$ (1)

Equation or Formula (1) describes the relationship between the petrol needed and distance travelled.

Step 2 : Solution : We want to find the petrol we need to travel 180 kilometres; so, we have to find the value of y when $x = 180$. Putting $x = 180$ in (1), we have

$$y = \frac{180}{9} = 20.$$

Step 3 : Interpretation : Since $y = 20$, we need 20 litres of petrol to travel 180 kilometres.

Did it occur to you that you may not be able to use the formula (1) in all situations? For example, suppose the 432 kilometres route is through mountains and the 180 kilometres route is through flat plains. The car will use up petrol at a faster rate in the first route, so we cannot use the same rate for the 180 kilometres route, where the petrol will be used up at a slower rate. So the formula works if all such conditions that affect the rate at which petrol is used are the same in both the trips. Or, if there is a difference in conditions, the effect of the difference on the amount of petrol needed for the car should be very small. The petrol used will vary directly with the distance travelled only in such a situation. We assumed this while solving the problem.

Example 2 : Suppose Sudhir has invested ₹ 15,000 at 8% simple interest per year. With the return from the investment, he wants to buy a washing machine that costs ₹ 19,000. For what period should he invest ₹ 15,000 so that he has enough money to buy a washing machine?

Solution : Step 1 : Formulation of the problem : Here, we know the principal and the rate of interest. The interest is the amount Sudhir needs in addition to 15,000 to buy the washing machine. We have to find the number of years.

Mathematical Description : The formula for simple interest is $I = \frac{Pnr}{100}$,

where

P = Principal,

n = Number of years,

r % = Rate of interest

I = Interest earned

Here,

the principal = ₹ 15,000

The money required by Sudhir for buying a washing machine = ₹ 19,000

So, the interest to be earned = ₹ (19,000 – 15,000)
= ₹ 4,000

The number of years for which ₹ 15,000 is deposited = n

The interest on ₹ 15,000 for n years at the rate of 8% = I

Then,

$$I = \frac{15000 \times n \times 8}{100}$$

So, $I = 1200n$ (1)

gives the relationship between the number of years and interest, if ₹ 15000 is invested at an annual interest rate of 8%.

We have to find the period in which the interest earned is ₹ 4000. Putting $I = 4000$ in (1), we have

$$4000 = 1200n \quad (2)$$

Step 2 : Solution of the problem : Solving Equation (2), we get

$$n = \frac{4000}{1200} = 3\frac{1}{3}$$

Step 3 : Interpretation : Since $n = 3\frac{1}{3}$ and one third of a year is 4 months, Sudhir can buy a washing machine after 3 years and 4 months.

Can you guess the assumptions that you have to make in the example above? We have to assume that the interest rate remains the same for the period for which we calculate the interest. Otherwise, the formula $I = \frac{Pnr}{100}$ will not be valid. We have also assumed that the price of the washing machine does not increase by the time Sudhir has gathered the money.

Example 3 : A motorboat goes upstream on a river and covers the distance between two towns on the riverbank in six hours. It covers this distance downstream in five hours. If the speed of the stream is 2 km/h, find the speed of the boat in still water.

Solution : Step 1 : Formulation : We know the speed of the river and the time taken to cover the distance between two places. We have to find the speed of the boat in still water.

Mathematical Description : Let us write x for the speed of the boat, t for the time taken and y for the distance travelled. Then

$$y = tx \quad (1)$$

Let d be the distance between the two places.

While going upstream, the actual speed of the boat

$$= \text{speed of the boat} - \text{speed of the river,}$$

because the boat is travelling against the flow of the river.

So, the speed of the boat upstream $= (x - 2)$ km/h

It takes 6 hours to cover the distance between the towns upstream. So, from (1),

we get $d = 6(x - 2)$ (2)

When going downstream, the speed of the river has to be *added* to the speed of the boat.

So, the speed of the boat downstream = $(x + 2)$ km/h

The boat takes 5 hours to cover the same distance downstream. So,

$$d = 5(x + 2) \quad (3)$$

From (2) and (3), we have

$$5(x + 2) = 6(x - 2) \quad (4)$$

Step 2 : Finding the Solution

Solving for x in Equation (4), we get $x = 22$.

Step 3 : Interpretation

Since $x = 22$, therefore the speed of the motorboat in still water is 22 km/h.

In the example above, we know that the speed of the river is not the same everywhere. It flows slowly near the shore and faster at the middle. The boat starts at the shore and moves to the middle of the river. When it is close to the destination, it will slow down and move closer to the shore. So, there is a small difference between the speed of the boat at the middle and the speed at the shore. Since it will be close to the shore for a small amount of time, this difference in speed of the river will affect the speed only for a small period. So, we can ignore this difference in the speed of the river. We can also ignore the small variations in speed of the boat. Also, apart from the speed of the river, the friction between the water and surface of the boat will also affect the actual speed of the boat. We also assume that this effect is very small.

So, we have assumed that

1. The speed of the river and the boat remains constant all the time.
2. The effect of friction between the boat and water and the friction due to air is negligible.

We have found the speed of the boat in still water with the *assumptions (hypotheses)* above.

As we have seen in the word problems above, there are 3 steps in solving a word problem. These are

1. **Formulation :** We analyse the problem and see which factors have a major influence on the solution to the problem. These are the **relevant factors**. In our first example, the relevant factors are the distance travelled and petrol consumed. We ignored the other factors like the nature of the route, driving speed, etc. Otherwise, the problem would have been more difficult to solve. The factors that we ignore are the **irrelevant factors**.

We then describe the problem mathematically, in the form of one or more mathematical equations.

2. **Solution :** We find the solution of the problem by solving the mathematical equations obtained in Step 1 using some suitable method.
3. **Interpretation :** We see what the solution obtained in Step 2 means in the context of the original word problem.

Here are some exercises for you. You may like to check your understanding of the steps involved in solving word problems by carrying out the three steps above for the following problems.

EXERCISE A2.1

In each of the following problems, clearly state what the relevant and irrelevant factors are while going through Steps 1, 2 and 3 given above.

1. Suppose a company needs a computer for some period of time. The company can either hire a computer for ₹ 2,000 per month or buy one for ₹ 25,000. If the company has to use the computer for a long period, the company will pay such a high rent, that buying a computer will be cheaper. On the other hand, if the company has to use the computer for say, just one month, then hiring a computer will be cheaper. Find the number of months beyond which it will be cheaper to buy a computer.
2. Suppose a car starts from a place A and travels at a speed of 40 km/h towards another place B. At the same instance, another car starts from B and travels towards A at a speed of 30 km/h. If the distance between A and B is 100 km, after how much time will the cars meet?
3. The moon is about 3,84,000 km from the earth, and its path around the earth is nearly circular. Find the speed at which it orbits the earth, assuming that it orbits the earth in 24 hours. (Use $\pi = 3.14$)
4. A family pays ₹ 1000 for electricity on an average in those months in which it does not use a water heater. In the months in which it uses a water heater, the average electricity bill is ₹ 1240. The cost of using the water heater is ₹ 8.00 per hour. Find the average number of hours the water heater is used in a day.

A2.3 Some Mathematical Models

So far, nothing was new in our discussion. In this section, we are going to add another step to the three steps that we have discussed earlier. This step is called *validation*. What does validation mean? Let us see. In a real-life situation, we cannot accept a model that gives us an answer that does not match the reality. This process of checking the answer against reality, and modifying the mathematical description if necessary, is

called *validation*. This is a very important step in modelling. We will introduce you to this step in this section.

First, let us look at an example, where we do not have to modify our model after validation.

Example 4 : Suppose you have a room of length 6 m and breadth 5 m. You want to cover the floor of the room with square mosaic tiles of side 30 cm. How many tiles will you need? Solve this by constructing a mathematical model.

Solution : Formulation : We have to consider the area of the room and the area of a tile for solving the problem. The side of the tile is 0.3 m. Since the length is 6 m, we

can fit in $\frac{6}{0.3} = 20$ tiles along the length of the room in one row (see Fig. A2.1.).

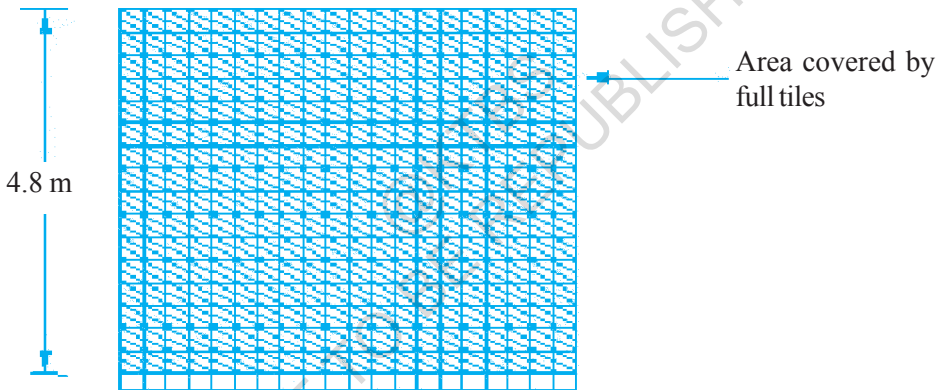


Fig. A2.1

Since the breadth of the room is 5 metres, we have $\frac{5}{0.3} = 16.67$. So, we can fit in 16 tiles in a column. Since $16 \times 0.3 = 4.8$, $5 - 4.8 = 0.2$ metres along the breadth will not be covered by tiles. This part will have to be covered by cutting the other tiles. The breadth of the floor left uncovered, 0.2 metres, is more than half the length of a tile, which is 0.3 m. So we cannot break a tile into two equal halves and use both the halves to cover the remaining portion.

Mathematical Description : We have:

Total number of tiles required = (Number of tiles along the length

× Number of tiles along the breadth) + Number of tiles along the uncovered area
(1)

Solution : As we said above, the number of tiles along the length is 20 and the number of tiles along the breadth is 16. We need 20 more tiles for the last row. Substituting these values in (1), we get $(20 \times 16) + 20 = 320 + 20 = 340$.

Interpretation : We need 340 tiles to cover the floor.

Validation : In real-life, your mason may ask you to buy some extra tiles to replace those that get damaged while cutting them to size. This number will of course depend upon the skill of your mason! But, we need not modify Equation (1) for this. This gives you a rough idea of the number of tiles required. So, we can stop here.

Let us now look at another situation now.

Example 5 : In the year 2000, 191 member countries of the U.N. signed a declaration. In this declaration, the countries agreed to achieve certain development goals by the year 2015. These are called the *millennium development goals*. One of these goals is to promote gender equality. One indicator for deciding whether this goal has been achieved is the ratio of girls to boys in primary, secondary and tertiary education. India, as a signatory to the declaration, is committed to improve this ratio. The data for the percentage of girls who are enrolled in primary schools is given in Table A2.1.

Table A2.1

Year	Enrolment (in %)
1991-92	41.9
1992-93	42.6
1993-94	42.7
1994-95	42.9
1995-96	43.1
1996-97	43.2
1997-98	43.5
1998-99	43.5
1999-2000	43.6*
2000-01	43.7*
2001-02	44.1*

Source : Educational statistics, webpage of Department of Education, GOI.

* indicates that the data is provisional.

Using this data, mathematically describe the rate at which the proportion of girls enrolled in primary schools grew. Also, estimate the year by which the enrolment of girls will reach 50%.

Solution : Let us first convert the problem into a mathematical problem.

Step 1 : Formulation : Table A2.1 gives the enrolment for the years 1991-92, 1992-93, etc. Since the students join at the beginning of an academic year, we can take the years as 1991, 1992, etc. Let us assume that the percentage of girls who join primary schools will continue to grow at the same rate as the rate in Table A2.1. So, the number of years is important, not the specific years. (To give a similar situation, when we find the simple interest for, say, ₹ 1500 at the rate of 8% for three years, it does not matter whether the three-year period is from 1999 to 2002 or from 2001 to 2004. What is important is the interest rate in the years being considered). Here also, we will see how the enrolment grows after 1991 by comparing the number of years that has passed after 1991 and the enrolment. Let us take 1991 as the 0th year, and write 1 for 1992 since 1 year has passed in 1992 after 1991. Similarly, we will write 2 for 1993, 3 for 1994, etc. So, Table A2.1 will now look like as Table A2.2.

Table A2.2

Year	Enrolment (in %)
0	41.9
1	42.6
2	42.7
3	42.9
4	43.1
5	43.2
6	43.5
7	43.5
8	43.6
9	43.7
10	44.1

The increase in enrolment is given in the following table :

Table A2.3

Year	Enrolment (in %)	Increase
0	41.9	0
1	42.6	0.7
2	42.7	0.1
3	42.9	0.2
4	43.1	0.2
5	43.2	0.1
6	43.5	0.3
7	43.5	0
8	43.6	0.1
9	43.7	0.1
10	44.1	0.4

At the end of the one-year period from 1991 to 1992, the enrolment has increased by 0.7% from 41.9% to 42.6%. At the end of the second year, this has increased by 0.1%, from 42.6% to 42.7%. From the table above, we cannot find a definite relationship between the number of years and percentage. But the increase is fairly steady. Only in the first year and in the 10th year there is a jump. The mean of the values is

$$\frac{0.7 + 0.1 + 0.2 + 0.2 + 0.1 + 0.3 + 0 + 0.1 + 0.1 + 0.4}{10} = 0.22$$

Let us assume that the enrolment steadily increases at the rate of 0.22 per cent.

Mathematical Description : We have assumed that the enrolment increases steadily at the rate of 0.22% per year.

So, the Enrolment Percentage (EP) in the first year = $41.9 + 0.22$

EP in the second year = $41.9 + 0.22 + 0.22 = 41.9 + 2 \times 0.22$

EP in the third year = $41.9 + 0.22 + 0.22 + 0.22 = 41.9 + 3 \times 0.22$

So, the enrolment percentage in the n th year = $41.9 + 0.22n$, for $n \geq 1$. (1)

Now, we also have to find the number of years by which the enrolment will reach 50%. So, we have to find the value of n in the equation or formula

$$50 = 41.9 + 0.22n \quad (2)$$

Step 2 : Solution : Solving (2) for n , we get

$$n = \frac{50 - 41.9}{0.22} = \frac{8.1}{0.22} = 36.8$$

Step 3 : Interpretation : Since the number of years is an integral value, we will take the next higher integer, 37. So, the enrolment percentage will reach 50% in $1991 + 37 = 2028$.

In a word problem, we generally stop here. But, since we are dealing with a real-life situation, we have to see to what extent this value matches the real situation.

Step 4 : Validation: Let us check if Formula (2) is in agreement with the reality. Let us find the values for the years we already know, using Formula (2), and compare it with the known values by finding the difference. The values are given in Table A2.4.

Table A2.4

Year	Enrolment (in %)	Values given by (2) (in %)	Difference (in %)
0	41.9	41.90	0
1	42.6	42.12	0.48
2	42.7	42.34	0.36
3	42.9	42.56	0.34
4	43.1	42.78	0.32
5	43.2	43.00	0.20
6	43.5	43.22	0.28
7	43.5	43.44	0.06
8	43.6	43.66	-0.06
9	43.7	43.88	-0.18
10	44.1	44.10	0.00

As you can see, some of the values given by Formula (2) are less than the actual values by about 0.3% or even by 0.5%. This can give rise to a difference of about 3 to 5 years since the increase per year is actually 1% to 2%. We may decide that this

much of a difference is acceptable and stop here. In this case, (2) is our mathematical model.

Suppose we decide that this error is quite large, and we have to improve this model. Then we have to go back to Step 1, the formulation, and change Equation (2). Let us do so.

Step 1 : Reformulation : We still assume that the values increase steadily by 0.22%, but we will now introduce a correction factor to reduce the error. For this, we find the mean of all the errors. This is

$$\frac{0 + 0.48 + 0.36 + 0.34 + 0.32 + 0.2 + 0.28 + 0.06 - 0.06 - 0.18 + 0}{10} = 0.18$$

We take the mean of the errors, and correct our formula by this value.

Revised Mathematical Description : Let us now add the mean of the errors to our formula for enrolment percentage given in (2). So, our corrected formula is:

$$\text{Enrolment percentage in the } n\text{th year} = 41.9 + 0.22n + 0.18 = 42.08 + 0.22n, \text{ for } n \geq 1 \quad (3)$$

We will also modify Equation (2) appropriately. The new equation for n is:

$$50 = 42.08 + 0.22n \quad (4)$$

Step 2 : Altered Solution : Solving Equation (4) for n , we get

$$n = \frac{50 - 42.08}{0.22} = \frac{7.92}{0.22} = 36$$

Step 3 : Interpretation: Since $n = 36$, the enrolment of girls in primary schools will reach 50% in the year $1991 + 36 = 2027$.

Step 4 : Validation: Once again, let us compare the values got by using Formula (4) with the actual values. Table A2.5 gives the comparison.

Table A2.5

Year	Enrolment (in %)	Values given by (2)	Difference between values	Values given by (4)	Difference between values
0	41.9	41.90	0	41.9	0
1	42.6	42.12	0.48	42.3	0.3
2	42.7	42.34	0.36	42.52	0.18
3	42.9	42.56	0.34	42.74	0.16
4	43.1	42.78	0.32	42.96	0.14
5	43.2	43.00	0.2	43.18	0.02
6	43.5	43.22	0.28	43.4	0.1
7	43.5	43.44	0.06	43.62	− 0.12
8	43.6	43.66	− 0.06	43.84	− 0.24
9	43.7	43.88	− 0.18	44.06	− 0.36
10	44.1	44.10	0	44.28	− 0.18

As you can see, many of the values that (4) gives are closer to the actual value than the values that (2) gives. The mean of the errors is 0 in this case.

We will stop our process here. So, Equation (4) is our mathematical description that gives a mathematical relationship between years and the percentage of enrolment of girls of the total enrolment. We have constructed a mathematical model that describes the growth.

The process that we have followed in the situation above is called mathematical modelling.

We have tried to construct a mathematical model with the mathematical tools that we already have. There are better mathematical tools for making predictions from the data we have. But, they are beyond the scope of this course. Our aim in constructing this model is to explain the process of modelling to you, not to make accurate predictions at this stage.

You may now like to model some real-life situations to check your understanding of our discussion so far. Here is an Exercise for you to try.

EXERCISE A2.2

1. We have given the timings of the gold medalists in the 400-metre race from the time the event was included in the Olympics, in the table below. Construct a mathematical model relating the years and timings. Use it to estimate the timing in the next Olympics.

Table A2.6

Year	Timing (in seconds)
1964	52.01
1968	52.03
1972	51.08
1976	49.28
1980	48.88
1984	48.83
1988	48.65
1992	48.83
1996	48.25
2000	49.11
2004	49.41

A2.4 The Process of Modelling, its Advantages and Limitations

Let us now conclude our discussion by drawing out aspects of mathematical modelling that show up in the examples we have discussed. With the background of the earlier sections, we are now in a position to give a brief overview of the steps involved in modelling.

Step 1 : Formulation : You would have noticed the difference between the formulation part of Example 1 in Section A2.2 and the formulation part of the model we discussed in A2.3. In Example 1, all the information is in a readily usable form. But, in the model given in A2.3 this is not so. Further, it took us some time to find a mathematical description. We tested our first formula, but found that it was not as good as the second one we got. This is usually true in general, i.e. when trying to model real-life situations; the first model usually needs to be revised. When we are solving a real-life problem, formulation can require a lot of time. For example, Newton's three laws of motion, which are mathematical descriptions of motion, are simple enough to state. But, Newton arrived at these laws after studying a large amount of data and the work the scientists before him had done.

Formulation involves the following three steps :

- (i) **Stating the problem** : Often, the problem is stated vaguely. For example, the broad goal is to ensure that the enrolment of boys and girls are equal. This may mean that 50% of the total number of boys of the school-going age and 50% of the girls of the school-going age should be enrolled. The other way is to ensure that 50% of the school-going children are girls. In our problem, we have used the second approach.
- (ii) **Identifying relevant factors** : Decide which quantities and relationships are important for our problem and which are unimportant and can be neglected. For example, in our problem regarding primary schools enrolment, the percentage of girls enrolled in the previous year can influence the number of girls enrolled this year. This is because, as more and more girls enrol in schools, many more parents will feel they also have to put their daughters in schools. But, we have ignored this factor because this may become important only after the enrolment crosses a certain percentage. Also, adding this factor may make our model more complicated.
- (iii) **Mathematical Description** : Now suppose we are clear about what the problem is and what aspects of it are more relevant than the others. Then we have to find a relationship between the aspects involved in the form of an equation, a graph or any other suitable mathematical description. If it is an equation, then every important aspect should be represented by a variable in our mathematical equation.

Step 2 : Finding the solution : The mathematical formulation does not give the solution. We have to solve this mathematical equivalent of the problem. This is where your mathematical knowledge comes in useful.

Step 3 : Interpreting the solution : The mathematical solution is some value or values of the variables in the model. We have to go back to the real-life problem and see what these values mean in the problem.

Step 4 : Validating the solution : As we saw in A2.3, after finding the solution we will have to check whether the solution matches the reality. If it matches, then the mathematical model is acceptable. If the mathematical solution does not match, *we go back to the formulation step* again and try to improve our model.

This step in the process is one major difference between solving word problems and mathematical modelling. This is one of the most important step in modelling that is missing in word problems. Of course, it is possible that in some real-life situations, we do not need to validate our answer because the problem is simple and we get the correct solution right away. This was so in the first model we considered in A2.3.

We have given a summary of the order in which the steps in mathematical modelling are carried out in Fig. A2.2 below. Movement from the validation step to the formulation step is shown using a **dotted arrow**. This is because it may not be necessary to carry out this step again.

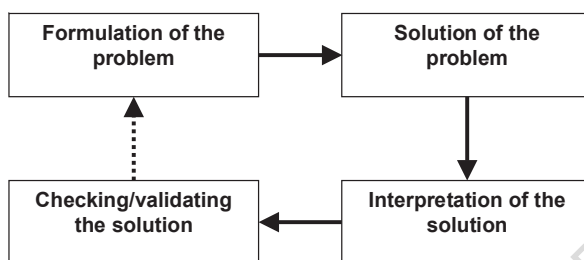


Fig.A2.2

Now that you have studied the stages involved in mathematical modelling, let us discuss some of its aspects.

The *aim* of mathematical modelling is to get some useful information about a real-world problem by converting it into a mathematical problem. This is especially useful when it is not possible or very expensive to get information by other means such as direct observation or by conducting experiments.

You may also wonder why we should undertake mathematical modelling? Let us look at some **advantages of modelling**. Suppose we want to study the corrosive effect of the discharge of the Mathura refinery on the Taj Mahal. We would not like to carry out experiments on the Taj Mahal directly since it may not be safe to do so. Of course, we can use a scaled down physical model, but we may need special facilities for this, which may be expensive. Here is where mathematical modelling can be of great use.

Again, suppose we want to know how many primary schools we will need after 5 years. Then, we can only solve this problem by using a mathematical model. Similarly, it is only through modelling that scientists have been able to explain the existence of so many phenomena.

You saw in Section A2.3, that we could have tried to improve the answer in the second example with better methods. But we stopped because we do not have the mathematical tools. This can happen in real-life also. Often, we have to be satisfied with very approximate answers, because mathematical tools are not available. For example, the model equations used in modelling weather are so complex that mathematical tools to find exact solutions are not available.

You may wonder to what extent we should try to improve our model. Usually, to improve it, we need to take into account more factors. When we do this, we add more variables to our mathematical equations. We may then have a very complicated model that is difficult to use. A model must be simple enough to use. A good model balances two factors:

1. Accuracy, i.e., how close it is to reality.
2. Ease of use.

For example, Newton's laws of motion are very simple, but powerful enough to model many physical situations.

So, is mathematical modelling the answer to all our problems? Not quite! It has its limitations.

Thus, we should keep in mind that a model is *only a simplification* of a real-world problem, and the two are not the same. It is something like the difference between a map that gives the physical features of a country, and the country itself. We can find the height of a place above the sea level from this map, but we cannot find the characteristics of the people from it. So, we should use a model only for the purpose it is supposed to serve, remembering all the factors we have neglected while constructing it. We should apply the model only within the limits where it is applicable. In the later classes, we shall discuss this aspect a little more.

EXERCISE A2.3

1. How are the solving of word problems that you come across in textbooks different from the process of mathematical modelling?
2. Suppose you want to minimise the waiting time of vehicles at a traffic junction of four roads. Which of these factors are important and which are not?
 - (i) Price of petrol.
 - (ii) The rate at which the vehicles arrive in the four different roads.
 - (iii) The proportion of slow-moving vehicles like cycles and rickshaws and fast moving vehicles like cars and motorcycles.

A2.5 Summary

In this Appendix, you have studied the following points :

1. The steps involved in solving word problems.
2. Construction of some mathematical models.

3. The steps involved in mathematical modelling given in the box below.

1. **Formulation :**
 - (i) Stating the question
 - (ii) Identifying the relevant factors
 - (iii) Mathematical description
2. **Finding the solution.**
3. **Interpretation of the solution in the context of the real-world problem.**
4. **Checking/validating to what extent the model is a good representation of the problem being studied.**

4. The aims, advantages and limitations of mathematical modelling.

ANSWERS/HINTS

EXERCISE 7.1

1. They are equal. 6. $\angle BAC = \angle DAE$

EXERCISE 7.2

6. $\angle BCD = \angle BCA + \angle DCA = \angle B + \angle D$ 7. each is of 45°

EXERCISE 7.3

3. (ii) From (i), $\angle ABM = \angle PQN$

EXERCISE 8.1

3. (i) From $\triangle DAC$ and $\triangle BCA$, show $\angle DAC = \angle BCA$ and $\angle ACD = \angle CAB$, etc.
(ii) Show $\angle BAC = \angle BCA$, using Theorem 8.4.

EXERCISE 8.2

2. Show PQRS is a parallelogram. Also show $PQ \parallel AC$ and $PS \parallel BD$. So, $\angle P = 90^\circ$.
5. AEFC is a parallelogram. So, $AF \parallel CE$, etc.

EXERCISE 9.1

1. Prove exactly as Theorem 9.1 by considering chords of congruent circles.
2. Use SAS axiom of congruence to show the congruence of the two triangles.

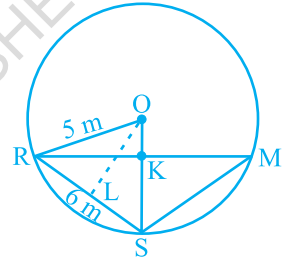
EXERCISE 9.2

1. 6 cm. First show that the line joining centres is perpendicular to the radius of the smaller circle and then that common chord is the diameter of the smaller circle.
2. If AB, CD are equal chords of a circle with centre O intersecting at E, draw perpendiculars OM on AB and ON on CD and join OE. Show that right triangles OME and ONE are congruent.
3. Proceed as in Example 2.
4. Draw perpendicular OM on AD.
5. Represent Reshma, Salma and Mandip by R, S and M respectively. Let KR = x m (see figure).

Area of $\triangle ORS = \frac{1}{2}x \times 5$. Also, area of $\triangle ORS =$

$$\frac{1}{2}RS \times OL = \frac{1}{2} \times 6 \times 4.$$

Find x and hence RM.



6. Use the properties of an equilateral triangle and also Pythagoras Theorem.

EXERCISE 9.3

1. 45°
2. $150^\circ, 30^\circ$
3. 10°
4. 80°
5. 110°
6. $\angle BCD = 80^\circ$ and $\angle ECD = 50^\circ$
8. Draw perpendiculars AM and BN on CD ($AB \parallel CD$ and $AB < CD$). Show $\triangle AMD \cong \triangle BNC$. This gives $\angle C = \angle D$ and, therefore, $\angle A + \angle C = 180^\circ$.

EXERCISE 10.1

1. $\frac{\sqrt{3}}{4}a^2, 900, 3\text{ cm}^2$
2. ₹1650000
3. $20\sqrt{2}\text{ m}^2$
4. $21\sqrt{11}\text{ cm}^2$
5. 9000 cm^2
6. $9\sqrt{15}\text{ cm}^2$

EXERCISE 11.1

1. 165 cm^2
2. 1244.57 m^2
3. (i) 7 cm (ii) 462 cm^2
4. (i) 26 m (ii) ₹ 137280
5. 63 m
6. ₹ 1155
7. 5500 cm^2
8. ₹ 384.34 (approx.)

EXERCISE 11.2

1. (i) 1386 cm^2 (ii) 394.24 cm^2 (iii) 2464 cm^2
2. (i) 616 cm^2 (ii) 1386 cm^2 (iii) 38.5 m^2
3. 942 cm^2
4. $1 : 4$
5. ₹ 27.72
6. 3.5 cm
7. $1 : 16$
8. 173.25 cm^2
9. (i) $4\pi r^2$ (ii) $4\pi r^2$ (iii) $1 : 1$

EXERCISE 11.3

1. (i) 264 cm^3 (ii) 154 cm^3
2. (i) 1.232 l (ii) $\frac{11}{35} \text{ l}$
3. 10 cm
4. 8 cm
5. 38.5 kl
6. (i) 48 cm (ii) 50 cm (iii) 2200 cm^2
7. $100\pi \text{ cm}^3$
8. $240\pi \text{ cm}^3; 5 : 12$
9. $86.625 \text{ m}^3, 99.825 \text{ m}^2$

EXERCISE 11.4

1. (i) $1437 \frac{1}{3} \text{ cm}^3$ (ii) 1.05 m^3 (approx.)
2. (i) $11498 \frac{2}{3} \text{ cm}^3$ (ii) 0.004851 m^3
3. 345.39 g (approx.)
4. $\frac{1}{64}$
5. 0.303 l (approx.)
6. 0.06348 m^3 (approx.)
7. $179 \frac{2}{3} \text{ cm}^3$
8. (i) 249.48 m^2 (ii) 523.9 m^3 (approx.)
9. (i) $3r$ (ii) $1 : 9$
10. 22.46 mm^3 (approx.)

EXERCISE 12.1

1. (ii) Reproductive health conditions.
 3. (ii) Party A 4. (ii) Frequency polygon (iii) No 5. (ii) 184

8.

Age (in years)	Frequency	Width	Length of the rectangle
1 - 2	5	1	$\frac{5}{1} \times 1 = 5$
2 - 3	3	1	$\frac{3}{1} \times 1 = 3$
3 - 5	6	2	$\frac{6}{2} \times 1 = 3$
5 - 7	12	2	$\frac{12}{2} \times 1 = 6$
7 - 10	9	3	$\frac{9}{3} \times 1 = 3$
10 - 15	10	5	$\frac{10}{5} \times 1 = 2$
15 - 17	4	2	$\frac{4}{2} \times 1 = 2$

Now, you can draw the histogram, using these lengths.

9. (i)

Number of letters	Frequency	Width of interval	Length of rectangle
1 - 4	6	3	$\frac{6}{3} \times 2 = 4$
4 - 6	30	2	$\frac{30}{2} \times 2 = 30$
6 - 8	44	2	$\frac{44}{2} \times 2 = 44$
8 - 12	16	4	$\frac{16}{4} \times 2 = 8$
12 - 20	4	8	$\frac{4}{8} \times 2 = 1$

Now, draw the histogram.

- (ii) 6 - 8

EXERCISE A2.1

1. Step 1: Formulation :

The relevant factors are the time period for hiring a computer, and the two costs given to us. We assume that there is no significant change in the cost of purchasing or hiring the computer. So, we treat any such change as irrelevant. We also treat all brands and generations of computers as the same, i.e. these differences are also irrelevant.

The expense of hiring the computer for x months is ₹ $2000x$. If this becomes more than the cost of purchasing a computer, we will be better off buying a computer. So, the equation is

$$2000x = 25000 \quad (1)$$

Step 2 : Solution : Solving (1), $x = \frac{25000}{2000} = 12.5$

Step 3 : Interpretation : Since the cost of hiring a computer becomes more **after** 12.5 months, it is cheaper to buy a computer, if you have to use it for more than 12 months.

- 2. Step 1 : Formulation :** We will assume that cars travel at a constant speed. So, any change of speed will be treated as irrelevant. If the cars meet after x hours, the first car would have travelled a distance of $40x$ km from A and the second car would have travelled $30x$ km, so that it will be at a distance of $(100 - 30x)$ km from A. So the equation will be $40x = 100 - 30x$, i.e., $70x = 100$.

Step 2 : Solution : Solving the equation, we get $x = \frac{100}{70}$.

Step 3 : Interpretation : $\frac{100}{70}$ is approximately 1.4 hours. So, the cars will meet after 1.4 hours.

- 3. Step 1: Formulation :** The speed at which the moon orbits the earth is

$$\frac{\text{Length of the orbit}}{\text{Time taken}}$$

Step 2 : Solution : Since the orbit is nearly circular, the length is $2 \times \pi \times 384000$ km = 2411520 km

The moon takes 24 hours to complete one orbit.

So, speed = $\frac{2411520}{24} = 100480$ km/hour.

Step 3 : Interpretation : The speed is 100480 km/h.

4. **Formulation :** An assumption is that the difference in the bill is only because of using the water heater.

Let the average number of hours for which the water heater is used = x

Difference per month due to using water heater = ₹ 1240 – ₹ 1000 = ₹ 240

Cost of using water heater for one hour = ₹ 8

So, the cost of using the water heater for 30 days = $8 \times 30 \times x$

Also, the cost of using the water heater for 30 days = Difference in bill due to using water heater

So, $240x = 240$

Solution : From this equation, we get $x = 1$.

Interpretation : Since $x = 1$, the water heater is used for an average of 1 hour in a day.

EXERCISE A2.2

1. We will not discuss any particular solution here. You can use the same method as we used in last example, or any other method you think is suitable.

EXERCISE A2.3

1. We have already mentioned that the formulation part could be very detailed in real-life situations. Also, we do not validate the answer in word problems. Apart from this word problem have a 'correct answer'. This need not be the case in real-life situations.
2. The important factors are (ii) and (iii). Here (i) is not an important factor although it can have an effect on the number of vehicles sold.